Homework #8
Introduction to physical oceanography

1. **Geostrophy**: Consider a zonal flow centered around the latitude $45°S$ with a width equal to $2L$. In the region $-L < y < L$, the velocity of the flow is defined by

$$u = u_0 \, cos\left(\frac{\pi y}{2L}\right) e^{z/H} \tag{1}$$

where $u_0 = 1m/s$, $L = 50km$, $H = 500m$ and z is the depth (taking $z = 0$ to be the mean sea level). The flow is assumed to be in geostrophic and hydrostatic balance. Outside this domain, the velocity of the flow is 0.

   (a) Plot using Matlab the velocity u as function of y.

   (b) Find and plot (using Matlab) the sea surface elevation as function of y. Assume that the reference density is $\rho_0 = 1025kg/m^3$.

   (c) Derive an expression for the density difference $\Delta\rho = \rho(y,z) - \rho(0,z)$. Plot contours of $\Delta\rho$ using the function contourf in Matlab.

   (d) Assume that the density is given by the linear equation of state (assume that there is no contribution from the salinity). Find the temperature difference $\Delta T(z) = T(L,z) - T(-L,z)$. Estimate ΔT at a depth of $300m$. (Use the thermal coefficient from the table posted on the course homepage for the given ρ_0)

2. **Acceleration, Eulerian/ Lagrangian:** Consider an Eulerian flow $u = a(x+y), v = a(x+y)$ where a is a constant coefficient. Sketch what the flow looks like. If this flow is due to a channel changing its width, what would the channel orientation be, and what is width of the channel as a function of the long-channel distance? Find the acceleration of fluid particles in the x and y directions, expressed in terms of Eulerian quantities, and in terms of Lagrangian quantities.

3. **Continuity equation** Show that the continuity equation for a layer of water of constant density over a flat bottom is

$$h_t + (uh)_x + (vh)_y = 0$$

where $h(x,y,t)$ is total water depth from surface to bottom. Hint: consider a small domain between x and $x+dx$ and between y and $y+dy$. The mass flow into this domain is balanced by an increase in the surface height.

4. **Width of western boundary current:** We showed in class that the meridional velocity, v decays away from the western boundary exponentially like $\exp(-(x-x_0)/(J/\beta))$.

   (a) What is the scale of the exponential decay?

   (b) Let us assume that this scale is approximately equal to the observe width of the Gulf Stream. Suppose that the Gulf Stream width at $30°N$ is equal to $50km$. Find the value of the friction coefficient necessary to explain this width.

(c) Our zonal momentum equations was $u_t - fv = -p_x/\rho_0 - Ju + \tau_{(x)}$. Suppose that the interior values of u and v are of the same order. Is the above value you found for the friction coefficient justifies the assumption that the friction term is much smaller than the Coriolis term? How much smaller is it?

5. **Planetary vs relative vorticity**: Please read section 4.2.1 "Vorticity" (p 85-90) in the Open University book ("ocean circulation" 2nd edition, see reading list on course home page). Answer questions 4.3 and 4.4 there. Note that "relative vorticity" is $\zeta \equiv \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$, and "planetary vorticity" is $f = 2\Omega \sin\theta$.

6. **Experimental; Ekman pumping:**

   (a) Make yourself a cup of tea. Stir it, take the spoon out and measure the time it takes the tea to stop spinning.

   (b) Next, calculate the time you would have expected this to happen: write a u-momentum equation that includes a balance between scale selective friction and linear acceleration. assume a $\sin(x/L)$ spatial structure where L is the width of the cup. Find the exponential decay time for the velocity based on the molecular viscosity of water. Is the decay time you calculated similar to the one you measured?

   [If you are curious: to obtain the correct decay time theoretically, it turns out that one needs to bring into account an Ekman layer that forms at the bottom of the cup and effectively dissipates the vorticity of the spinning tea.]

7. **Optional: Intuitive explanations for western boundary currents:** read Knauss around page 132 and Open university book around page 97. Try to reproduce their arguments for why the boundary current must be at the west. How is the open university explanation different from what we did in class. Can you explain the source of this difference?