1 Products and Sums

For these questions, use the lambda calculus with products and sums (Lecture 12, §1.1).

(a) Write a program that constructs two values of type \( \text{int} + (\text{int} \to \text{int}) \), one using left injection, and one using right injection.

(b) Write a function that takes a value of type \( \text{int} + (\text{int} \to \text{int}) \) and if the value is an integer, it adds 7 to it, and if the value is a function it applies the function to 42.

(c) Give a typing derivation for the following program.

\[
\lambda p : (\text{unit} \to \text{int}) \times (\text{int} \to \text{int}). \lambda x : \text{unit}. \text{case } x \text{ of } \#1 p | \#2 p
\]

(d) Write a program that uses the term in part (c) above to produce the value 42.

2 Recursion

(a) Use the \( x : e \) expression to write a function that takes a natural number \( n \) and returns the sum of all even natural numbers less than or equal to \( n \). (You can assume you have appropriate integer comparison operators, and also a modulus operator.)

(b) Try executing your program by applying it to the number 5.

(c) Give a typing derivation for the following program. What happens if you execute the program?

\[
\mu p : (\text{int} \to \text{int}) \times (\text{int} \to \text{int}). (\lambda n : \text{int}. n + 1, \#1 p)
\]

3 References

(a) Give a typing derivation for the following program.

\[
\text{let } a : \text{int} \text{ ref } = \text{ref } 4 \text{ in} \\
\text{let } b : \text{int} \text{ ref } = \text{ref } \lambda x : \text{int}. x + 38 \text{ in} \\
!b !a
\]

(b) Execute the program above for 4 small steps, to get configuration \( (e, \sigma) \). What is an appropriate \( \Sigma \) such that \( \emptyset, \Sigma \vdash e : \tau \) and \( \Sigma \vdash \sigma ? \)