

# Small-step Operational Semantics

CS 152 (Spring 2020)

Harvard University

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# Abstract Syntax

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$x, y, z \in \mathbf{Var}$

$n, m \in \mathbf{Int}$

$e \in \mathbf{Exp}$

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$x, y, z \in \mathbf{Var}$

**Var** is the set of program variables (e.g., foo, bar, baz, i, etc.).

# Abstract Syntax

$n, m \in \mathbf{Int}$

**Int** is the set of constant integers (e.g., 42, -40, 7).

# Abstract Syntax

$e \in \mathbf{Exp}$

**Exp** is the domain of expressions, which we specify using a BNF (Backus-Naur Form) grammar.

# Expressions

$e ::= x$

|  $n$

|  $e_1 + e_2$

|  $e_1 \times e_2$

|  $x := e_1; e_2$

# Assignment

$$x := e_1; e_2$$

Informally, the expression  $x := e_1; e_2$  means that  $x$  is assigned the value of  $e_1$  before evaluating  $e_2$ . The result of the entire expression is that of  $e_2$ .



# Abstract Syntax Tree

$$1 + 2 \times 3$$

# Abstract Syntax Tree

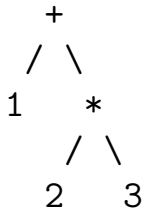
$$1 + 2 \times 3$$

$$1 + (2 \times 3)$$

$$(1 + 2) \times 3$$

# Abstract Syntax Tree

$1 + 2 \times 3$

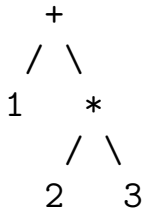


$1 + (2 \times 3)$

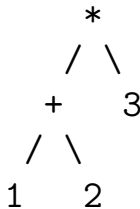
$(1 + 2) \times 3$

# Abstract Syntax Tree

$1 + 2 \times 3$



$1 + (2 \times 3)$



$(1 + 2) \times 3$

# Semantics

# Styles of Semantics

Operational Semantics

Denotational Semantics

Axiomatic Semantics

Algebraic Semantics

# Operational Semantics

Small-Step

Large-Step

# Small-Step Operational Semantics



# Configuration (State of Machine)

**Config = Exp  $\times$  Store**

**Store = Var  $\rightarrow$  Int**

# Step Relation

$$\longrightarrow \subseteq \mathbf{Config} \times \mathbf{Config}$$

# Notation

$$(\langle e_1, \sigma_1 \rangle, \langle e_2, \sigma_2 \rangle) \in \longrightarrow$$

$$\langle e_1, \sigma_1 \rangle \longrightarrow \langle e_2, \sigma_2 \rangle$$

# Rules Var

$$\text{VAR} \frac{\text{where } n = \sigma(x)}{\langle x, \sigma \rangle \longrightarrow \langle n, \sigma \rangle}$$

# Rules Add

$$\text{LADD} \frac{\langle e_1, \sigma \rangle \longrightarrow \langle e'_1, \sigma' \rangle}{\langle e_1 + e_2, \sigma \rangle \longrightarrow \langle e'_1 + e_2, \sigma' \rangle}$$

$$\text{RADD} \frac{\langle e_2, \sigma \rangle \longrightarrow \langle e'_2, \sigma' \rangle}{\langle n + e_2, \sigma \rangle \longrightarrow \langle n + e'_2, \sigma' \rangle}$$

$$\text{ADD} \frac{\text{where } p \text{ is the sum of } n \text{ and } m}{\langle n + m, \sigma \rangle \longrightarrow \langle p, \sigma \rangle}$$

# Rules Mul

$$\text{LMUL} \frac{\langle e_1, \sigma \rangle \longrightarrow \langle e'_1, \sigma' \rangle}{\langle e_1 \times e_2, \sigma \rangle \longrightarrow \langle e'_1 \times e_2, \sigma' \rangle}$$

$$\text{RMUL} \frac{\langle e_2, \sigma \rangle \longrightarrow \langle e'_2, \sigma' \rangle}{\langle n \times e_2, \sigma \rangle \longrightarrow \langle n \times e'_2, \sigma' \rangle}$$

$$\text{MUL} \frac{\text{where } p \text{ is the product of } n \text{ and } m}{\langle n \times m, \sigma \rangle \longrightarrow \langle p, \sigma \rangle}$$

# Rules Asg

$$\text{ASG1} \frac{\langle e_1, \sigma \rangle \longrightarrow \langle e'_1, \sigma' \rangle}{\langle x := e_1; e_2, \sigma \rangle \longrightarrow \langle x := e'_1; e_2, \sigma' \rangle}$$

$$\text{ASG} \frac{}{\langle x := n; e_2, \sigma \rangle \longrightarrow \langle e_2, \sigma[x \mapsto n] \rangle}$$

## Store Update $\sigma[x \mapsto n]$

If  $f$  is the function  $\sigma[x \mapsto n]$ , then

$$f(y) = \begin{cases} n & \text{if } y = x \\ \sigma(y) & \text{otherwise} \end{cases}$$



# Semantic Rules (1/2)

$$\text{LADD} \frac{\langle e_1, \sigma \rangle \longrightarrow \langle e'_1, \sigma' \rangle}{\langle e_1 + e_2, \sigma \rangle \longrightarrow \langle e'_1 + e_2, \sigma' \rangle}$$

$$\text{RADD} \frac{\langle e_2, \sigma \rangle \longrightarrow \langle e'_2, \sigma' \rangle}{\langle n + e_2, \sigma \rangle \longrightarrow \langle n + e'_2, \sigma' \rangle}$$

$$\text{ADD} \frac{}{\langle n + m, \sigma \rangle \longrightarrow \langle p, \sigma \rangle} \text{ where } p \text{ is the sum of } n \text{ and } m$$

$$\text{LMUL} \frac{\langle e_1, \sigma \rangle \longrightarrow \langle e'_1, \sigma' \rangle}{\langle e_1 \times e_2, \sigma \rangle \longrightarrow \langle e'_1 \times e_2, \sigma' \rangle}$$

$$\text{RMUL} \frac{\langle e_2, \sigma \rangle \longrightarrow \langle e'_2, \sigma' \rangle}{\langle n \times e_2, \sigma \rangle \longrightarrow \langle n \times e'_2, \sigma' \rangle}$$

$$\text{MUL} \frac{}{\langle n \times m, \sigma \rangle \longrightarrow \langle p, \sigma \rangle} \text{ where } p \text{ is the product of } n \text{ and } m$$

## Semantic Rules (2/2)

$$\text{VAR} \frac{}{\langle x, \sigma \rangle \longrightarrow \langle n, \sigma \rangle} \text{ where } n = \sigma(x)$$

$$\text{ASG1} \frac{\langle e_1, \sigma \rangle \longrightarrow \langle e'_1, \sigma' \rangle}{\langle x := e_1; e_2, \sigma \rangle \longrightarrow \langle x := e'_1; e_2, \sigma' \rangle}$$

$$\text{ASG} \frac{}{\langle x := n; e_2, \sigma \rangle \longrightarrow \langle e_2, \sigma[x \mapsto n] \rangle}$$

# Semantic Rules Recap

- ▶ LADD, RADD, ADD
- ▶ LMUL, RMUL, MUL
- ▶ VAR
- ▶ ASG1, ASG

# Semantic **Congruence** Rules Recap

- ▶ **LAdd**, **RAdd**, **ADD**
- ▶ **LMul**, **RMul**, **MUL**
- ▶ **VAR**
- ▶ **Asg1**, **ASG**

# Semantic **Computation** Rules Recap

- ▶ LADD, RADD, **Add**
- ▶ LMUL, RMUL, **Mul**
- ▶ **Var**
- ▶ ASG1, **Asg**

# Semantic Rules (Computation)

$$\text{VAR} \frac{}{\langle x, \sigma \rangle \longrightarrow \langle n, \sigma \rangle} \text{ where } n = \sigma(x)$$

$$\text{ADD} \frac{}{\langle n + m, \sigma \rangle \longrightarrow \langle p, \sigma \rangle} \text{ where } p \text{ is the sum of } n \text{ and } m$$

$$\text{MUL} \frac{}{\langle n \times m, \sigma \rangle \longrightarrow \langle p, \sigma \rangle} \text{ where } p \text{ is the product of } n \text{ and } m$$

$$\text{ASG} \frac{}{\langle x := n; e_2, \sigma \rangle \longrightarrow \langle e_2, \sigma[x \mapsto n] \rangle}$$

# Semantic Rules (Congruence)

$$\text{LADD} \frac{\langle e_1, \sigma \rangle \longrightarrow \langle e'_1, \sigma' \rangle}{\langle e_1 + e_2, \sigma \rangle \longrightarrow \langle e'_1 + e_2, \sigma' \rangle}$$

$$\text{RADD} \frac{\langle e_2, \sigma \rangle \longrightarrow \langle e'_2, \sigma' \rangle}{\langle n + e_2, \sigma \rangle \longrightarrow \langle n + e'_2, \sigma' \rangle}$$

$$\text{LMUL} \frac{\langle e_1, \sigma \rangle \longrightarrow \langle e'_1, \sigma' \rangle}{\langle e_1 \times e_2, \sigma \rangle \longrightarrow \langle e'_1 \times e_2, \sigma' \rangle}$$

$$\text{RMUL} \frac{\langle e_2, \sigma \rangle \longrightarrow \langle e'_2, \sigma' \rangle}{\langle n \times e_2, \sigma \rangle \longrightarrow \langle n \times e'_2, \sigma' \rangle}$$

$$\text{ASG1} \frac{\langle e_1, \sigma \rangle \longrightarrow \langle e'_1, \sigma' \rangle}{\langle x := e_1; e_2, \sigma \rangle \longrightarrow \langle x := e'_1; e_2, \sigma' \rangle}$$

# Using the Semantic Rules



# Using the Semantic Rules

$\langle (\text{foo} + 2) \times (\text{bar} + 1), \sigma \rangle$

where  $\sigma(\text{foo}) = 4$  and  $\sigma(\text{bar}) = 3$

# Using the Semantic Rules

$$\text{LM}_{\text{UL}} \frac{\langle \text{foo} + 2, \sigma \rangle \longrightarrow \langle e'_1, \sigma \rangle}{\langle (\text{foo} + 2) \times (\text{bar} + 1), \sigma \rangle \longrightarrow \langle e'_1 \times (\text{bar} + 1), \sigma \rangle}$$

# Using the Semantic Rules

$$\text{LADD} \frac{\langle \text{foo}, \sigma \rangle \longrightarrow \langle e_1'', \sigma \rangle}{\langle \text{foo} + 2, \sigma \rangle \longrightarrow \langle e_1'' + 2, \sigma \rangle}$$

# Using the Semantic Rules

$$\text{VAR} \frac{}{\langle \text{foo}, \sigma \rangle \longrightarrow \langle 4, \sigma \rangle}$$

# Using the Semantic Rules

$$\text{LMUL} \frac{\text{LADD} \frac{\text{VAR} \frac{}{\langle \text{foo}, \sigma \rangle \rightarrow \langle 4, \sigma \rangle}}{\langle \text{foo} + 2, \sigma \rangle \rightarrow \langle 4 + 2, \sigma \rangle}}{\langle (\text{foo} + 2) \times (\text{bar} + 1), \sigma \rangle \rightarrow \langle (4 + 2) \times (\text{bar} + 1), \sigma \rangle}}$$

Writing Derivation:

Bottom Left, Go Up, Top Right, Go Down

LMUL  $\frac{\quad}{\langle (\text{foo} + 2) \times (\text{bar} + 1), \sigma \rangle \rightarrow}$

# Writing Derivation:

Bottom Left, Go Up, Top Right, Go Down

$$\text{LMUL} \frac{\text{LADD} \frac{}{\langle \text{foo} + 2, \sigma \rangle \longrightarrow}}{\langle (\text{foo} + 2) \times (\text{bar} + 1), \sigma \rangle \longrightarrow}$$

# Writing Derivation:

Bottom Left, Go Up, Top Right, Go Down

$$\begin{array}{c} \text{VAR} \frac{\text{-----}}{\langle \text{foo}, \sigma \rangle \longrightarrow} \\ \text{LADD} \frac{\text{-----}}{\langle \text{foo} + 2, \sigma \rangle \longrightarrow} \\ \text{LMUL} \frac{\text{-----}}{\langle (\text{foo} + 2) \times (\text{bar} + 1), \sigma \rangle \longrightarrow} \end{array}$$



# Writing Derivation:

Bottom Left, Go Up, Top Right, Go Down

$$\begin{array}{c} \text{VAR} \frac{\text{---}}{\langle \text{foo}, \sigma \rangle \longrightarrow \langle 4, \sigma \rangle} \\ \text{LADD} \frac{\text{---}}{\langle \text{foo} + 2, \sigma \rangle \longrightarrow} \\ \text{LMUL} \frac{\text{---}}{\langle (\text{foo} + 2) \times (\text{bar} + 1), \sigma \rangle \longrightarrow} \end{array}$$

# Writing Derivation:

Bottom Left, Go Up, Top Right, Go Down

$$\begin{array}{c} \text{VAR} \frac{\text{---}}{\langle \text{foo}, \sigma \rangle \longrightarrow \langle 4, \sigma \rangle} \\ \text{LADD} \frac{\text{---}}{\langle \text{foo} + 2, \sigma \rangle \longrightarrow \langle 4 + 2, \sigma \rangle} \\ \text{LMUL} \frac{\text{---}}{\langle (\text{foo} + 2) \times (\text{bar} + 1), \sigma \rangle \longrightarrow} \end{array}$$

# Writing Derivation:

Bottom Left, Go Up, Top Right, Go Down

$$\text{LMUL} \frac{\text{LADD} \frac{\text{VAR} \frac{}{\langle \text{foo}, \sigma \rangle \longrightarrow \langle 4, \sigma \rangle}}{\langle \text{foo} + 2, \sigma \rangle \longrightarrow \langle 4 + 2, \sigma \rangle}}{\langle (\text{foo} + 2) \times (\text{bar} + 1), \sigma \rangle \longrightarrow \langle (4 + 2) \times (\text{bar} + 1), \sigma \rangle}}$$

## Using the Semantic Rules (Next Step)

$$\text{LMUL} \frac{\text{ADD} \frac{}{\langle 4 + 2, \sigma \rangle \longrightarrow \langle 6, \sigma \rangle}}{\langle (4 + 2) \times (\text{bar} + 1), \sigma \rangle \longrightarrow \langle 6 \times (\text{bar} + 1), \sigma \rangle}$$

# Using the Semantic Rules (All Steps)

$$\begin{aligned} & \langle (\text{foo} + 2) \times (\text{bar} + 1), \sigma \rangle \\ \longrightarrow & \langle (4 + 2) \times (\text{bar} + 1), \sigma \rangle \\ \longrightarrow & \langle 6 \times (\text{bar} + 1), \sigma \rangle \\ \longrightarrow & \langle 6 \times (3 + 1), \sigma \rangle \\ \longrightarrow & \langle 6 \times 4, \sigma \rangle \\ \longrightarrow & \langle 24, \sigma \rangle \end{aligned}$$

# Using the Semantic Rules (Multi-Steps)

$$\langle (\text{foo} + 2) \times (\text{bar} + 1), \sigma \rangle \longrightarrow^* \langle 24, \sigma \rangle.$$

# Expressing Program Properties

# Progress

$\forall e \in \mathbf{Exp}. \forall \sigma \in \mathbf{Store}.$

either  $e \in \mathbf{Int}$  or  $\exists e', \sigma'. \langle e, \sigma \rangle \longrightarrow \langle e', \sigma' \rangle$



# Termination

$$\forall e \in \mathbf{Exp}. \forall \sigma_0 \in \mathbf{Store}. \exists \sigma \in \mathbf{Store}. \exists n \in \mathbf{Int}. \\ \langle e, \sigma_0 \rangle \longrightarrow^* \langle n, \sigma \rangle$$

# Deterministic Result

$\forall e \in \mathbf{Exp}. \forall \sigma_0, \sigma, \sigma' \in \mathbf{Store}. \forall n, n' \in \mathbf{Int}.$   
if  $\langle e, \sigma_0 \rangle \longrightarrow^* \langle n, \sigma \rangle$  and  
 $\langle e, \sigma_0 \rangle \longrightarrow^* \langle n', \sigma' \rangle$  then  
 $n = n'$  and  $\sigma = \sigma'$ .

# Break

- ▶ Is `foo := 1` a valid expression?
- ▶ What about `foo := 1; bar := foo; bar`?  
What is the unambiguous abstract syntax?
- ▶ What about `bar := (foo := 1; foo); bar + 1`?
- ▶ What is the meaning of these expressions?