

Large-step semantics

CS 152 (Spring 2022)

Harvard University

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Today, we learn to

- ▶ define and use large-step operational semantics (also known as big-step semantics)
- ▶ prove equivalence between semantics (large-step and small-step)
- ▶ discuss the advantages of each style of semantics

Small-step semantics (Review)

$\longrightarrow \subseteq \mathbf{Config} \times \mathbf{Config}$

$\langle e, \sigma \rangle \longrightarrow^* \langle n, \sigma' \rangle$

Large-step semantics

$$\Downarrow \subseteq \mathbf{Config} \times \mathbf{FinalConfig}$$

where

$$\begin{aligned} \mathbf{Config} &= \mathbf{Exp} \times \mathbf{Store} \\ \text{and } \mathbf{FinalConfig} &= \mathbf{Int} \times \mathbf{Store} \subseteq \mathbf{Config}. \end{aligned}$$

We write $\langle e, \sigma \rangle \Downarrow \langle n, \sigma' \rangle$ to mean that
 $(\langle e, \sigma \rangle, \langle n, \sigma' \rangle) \in \Downarrow$

Small-step vs Large-step

$$\langle e, \sigma \rangle \Downarrow \langle n, \sigma' \rangle$$

$$\langle e, \sigma \rangle \longrightarrow^* \langle n, \sigma' \rangle$$

Small-step vs Large-step

$$\langle e, \sigma \rangle \Downarrow \langle n, \sigma' \rangle$$

$$\langle e, \sigma \rangle \longrightarrow^* \langle n, \sigma' \rangle$$

$$\langle e, \sigma \rangle \longrightarrow \langle e_1, \sigma_1 \rangle \dots \longrightarrow \langle e_k, \sigma_k \rangle \longrightarrow \langle n, \sigma' \rangle$$

Semantic Rules (1)

$$\text{INT}_{\text{LRG}} \frac{}{\langle n, \sigma \rangle \Downarrow \langle n, \sigma \rangle}$$

$$\text{VAR}_{\text{LRG}} \frac{}{\langle x, \sigma \rangle \Downarrow \langle n, \sigma \rangle} \text{ where } n = \sigma(x)$$

$$\text{ADD}_{\text{LRG}} \frac{\begin{array}{l} \langle e_1, \sigma \rangle \Downarrow \langle n_1, \sigma'' \rangle \\ \langle e_2, \sigma'' \rangle \Downarrow \langle n_2, \sigma' \rangle \\ \text{where } n \text{ is the sum of } n_1 \text{ and } n_2 \end{array}}{\langle e_1 + e_2, \sigma \rangle \Downarrow \langle n, \sigma' \rangle}$$

Semantic Rules (2)

$$\text{MUL}_{\text{LRG}} \frac{\begin{array}{l} \langle e_1, \sigma \rangle \Downarrow \langle n_1, \sigma'' \rangle \\ \langle e_2, \sigma'' \rangle \Downarrow \langle n_2, \sigma' \rangle \end{array} \text{ where } n \text{ is the product of } n_1 \text{ and } n_2}{\langle e_1 \times e_2, \sigma \rangle \Downarrow \langle n, \sigma' \rangle}$$

$$\text{ASG}_{\text{LRG}} \frac{\begin{array}{l} \langle e_1, \sigma \rangle \Downarrow \langle n_1, \sigma'' \rangle \\ \langle e_2, \sigma''[x \mapsto n_1] \rangle \Downarrow \langle n_2, \sigma' \rangle \end{array}}{\langle x := e_1; e_2, \sigma \rangle \Downarrow \langle n_2, \sigma' \rangle}$$

Example

$\langle \text{foo} := 3; \text{foo} \times \text{bar}, \sigma \rangle \Downarrow \langle 21, \sigma' \rangle$ for a store σ such that $\sigma(\text{bar}) = 7$, and $\sigma' = \sigma[\text{foo} \mapsto 3]$.

Example Proof Tree

$$\frac{\text{INT}_{\text{LRG}} \frac{}{\langle 3, \sigma \rangle \Downarrow \langle 3, \sigma \rangle} \quad \text{MUL}_{\text{LRG}} \frac{\text{VAR}_{\text{LRG}} \frac{}{\langle \text{foo}, \sigma' \rangle \Downarrow \langle 3, \sigma' \rangle} \quad \text{VAR}_{\text{LRG}} \frac{}{\langle \text{bar}, \sigma' \rangle \Downarrow \langle 7, \sigma' \rangle}}{\langle \text{foo} \times \text{bar}, \sigma' \rangle \Downarrow \langle 21, \sigma' \rangle}}{\langle \text{foo} := 3; \text{foo} \times \text{bar}, \sigma \rangle \Downarrow \langle 21, \sigma' \rangle}$$

Equivalence of semantics

For all expressions e , stores σ and σ' , and integers n , we have:

$$\langle e, \sigma \rangle \Downarrow \langle n, \sigma' \rangle \iff \langle e, \sigma \rangle \longrightarrow^* \langle n, \sigma' \rangle .$$



$$P(e) = \forall \sigma, \sigma' \in \mathbf{Store.}, \forall n \in \mathbf{Int.}$$

$$\langle e, \sigma \rangle \Downarrow \langle n, \sigma' \rangle$$



$$\langle e, \sigma \rangle \longrightarrow^* \langle n, \sigma' \rangle$$

Proof by structural induction on expression e .

\implies Case $e \equiv x$

We have

$$\langle x, \sigma \rangle \Downarrow \langle n, \sigma' \rangle$$

and need to show

$$\langle x, \sigma \rangle \longrightarrow^* \langle n, \sigma' \rangle$$

\implies Case $e \equiv x$

- ▶ By hypothesis, $\langle x, \sigma \rangle \Downarrow \langle n, \sigma' \rangle$.
- ▶ By inversion (only VAR_{LRG} applies),
 $n = \sigma(x)$ and $\sigma = \sigma'$.
- ▶ By VAR , $\langle x, \sigma \rangle \longrightarrow \langle n, \sigma \rangle$.
- ▶ We can conclude $\langle x, \sigma \rangle \longrightarrow^* \langle n, \sigma \rangle$.

\implies Case $e \equiv x$

$$\text{VAR}_{\text{LRG}} \frac{}{\langle x, \sigma \rangle \Downarrow \langle n, \sigma \rangle} \text{ where } n = \sigma(x)$$

$$\text{VAR} \frac{}{\langle x, \sigma \rangle \longrightarrow \langle n, \sigma \rangle} \text{ where } n = \sigma(x)$$

- ▶ By hypothesis, $\langle x, \sigma \rangle \Downarrow \langle n, \sigma' \rangle$.
- ▶ By inversion (only VAR_{LRG} applies), $n = \sigma(x)$ and $\sigma = \sigma'$.
- ▶ By VAR , $\langle x, \sigma \rangle \longrightarrow \langle n, \sigma \rangle$.
- ▶ We can conclude $\langle x, \sigma \rangle \longrightarrow^* \langle n, \sigma \rangle$.

\implies Case $e \equiv n$

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- ▶ By hypothesis and inversion, we know that

$$\langle n, \sigma \rangle \Downarrow \langle n, \sigma \rangle$$

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- ▶ By hypothesis and inversion, we know that

$$\langle n, \sigma \rangle \Downarrow \langle n, \sigma \rangle$$

- ▶ By reflexivity

$$\langle n, \sigma \rangle \longrightarrow^* \langle n, \sigma \rangle$$

\implies Case $e \equiv e_1 + e_2$

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$$P(e_1) = \forall n, \sigma, \sigma' : \quad \langle e_1, \sigma \rangle \Downarrow \langle n, \sigma' \rangle \implies \\ \langle e_1, \sigma \rangle \longrightarrow^* \langle n, \sigma' \rangle$$

$$P(e_2) = \forall n, \sigma, \sigma' : \quad \langle e_2, \sigma \rangle \Downarrow \langle n, \sigma' \rangle \implies \\ \langle e_2, \sigma \rangle \longrightarrow^* \langle n, \sigma' \rangle$$

$$P(e) = \forall n, \sigma, \sigma' : \quad \langle e_1 + e_2, \sigma \rangle \Downarrow \langle n, \sigma' \rangle \implies \\ \langle e_1 + e_2, \sigma \rangle \longrightarrow^* \langle n, \sigma' \rangle$$

Lemma 1 (Needed)

If $\langle e, \sigma \rangle \longrightarrow^* \langle n, \sigma' \rangle$ then for all n_1, e_2 the following hold.

▶ $\langle e + e_2, \sigma \rangle \longrightarrow^* \langle n + e_2, \sigma' \rangle$

▶ $\langle e \times e_2, \sigma \rangle \longrightarrow^* \langle n \times e_2, \sigma' \rangle$

▶ $\langle n_1 + e, \sigma \rangle \longrightarrow^* \langle n_1 + n, \sigma' \rangle$

▶ $\langle n_1 \times e, \sigma \rangle \longrightarrow^* \langle n_1 \times n, \sigma' \rangle$

\implies Case $e \equiv e_1 \times e_2$

\implies Case $e \equiv x := e_1; e_2$

Need lemma similar to lemma 1:

$\forall e \in \mathbf{Exp}. \forall \sigma, \sigma' \in \mathbf{Store}, \forall n \in \mathbf{Int}.$

if $\langle e, \sigma \rangle \longrightarrow^* \langle n, \sigma' \rangle$ then

$\forall x \in \mathbf{Var}. \forall e_2 \in \mathbf{Exp}.$

$\langle x := e; e_2, \sigma \rangle \longrightarrow^* \langle e_2, \sigma'[x \mapsto n] \rangle .$

Equivalence of semantics (Recall)

For all expressions e , stores σ and σ' , and integers n , we have:

$$\langle e, \sigma \rangle \Downarrow \langle n, \sigma' \rangle \iff \langle e, \sigma \rangle \longrightarrow^* \langle n, \sigma' \rangle .$$



Proof by mathematical induction on the number of steps $\langle e, \sigma \rangle \longrightarrow^* \langle n, \sigma' \rangle$.



$$P(k) = \forall e \in \mathbf{Exp}, \sigma, \sigma' \in \mathbf{Store.}, \forall n \in \mathbf{Int.}$$

$$\langle e, \sigma \rangle \Downarrow \langle n, \sigma' \rangle$$



$$\langle e, \sigma \rangle \longrightarrow^k \langle n, \sigma' \rangle$$

Proof by mathematical induction on the number of steps k .

⇐ Base case

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If $\langle e, \sigma \rangle \xrightarrow{*} \langle n, \sigma' \rangle$ in zero steps, then we must have $e \equiv n$ and $\sigma' = \sigma$. By INT_{LRG} ,
 $\langle n, \sigma \rangle \Downarrow \langle n, \sigma \rangle$.

⇐ Inductive case

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- ▶ Assume $\langle e, \sigma \rangle \rightarrow \langle e'', \sigma'' \rangle \rightarrow^* \langle n, \sigma' \rangle$.
- ▶ Inductive Hypothesis: $\langle e'', \sigma'' \rangle \Downarrow \langle n, \sigma' \rangle$.

Lemma 2 (Needed)

For all e, e', σ , and n , if $\langle e, \sigma \rangle \longrightarrow \langle e', \sigma'' \rangle$ and $\langle e', \sigma'' \rangle \Downarrow \langle n, \sigma' \rangle$, then $\langle e, \sigma \rangle \Downarrow \langle n, \sigma' \rangle$.

Lemma 1

If $\langle e, \sigma \rangle \longrightarrow^* \langle n, \sigma' \rangle$ then for all n_1, e_2 the following hold.

▶ $\langle e + e_2, \sigma \rangle \longrightarrow^* \langle n + e_2, \sigma' \rangle$

▶ $\langle e \times e_2, \sigma \rangle \longrightarrow^* \langle n \times e_2, \sigma' \rangle$

▶ $\langle n_1 + e, \sigma \rangle \longrightarrow^* \langle n_1 + n, \sigma' \rangle$

▶ $\langle n_1 \times e, \sigma \rangle \longrightarrow^* \langle n_1 \times n, \sigma' \rangle$

Proof of Lemma 1

By mathematical induction on the number of evaluation steps in \longrightarrow^* .

Proof of Lemma 1 (Base case)

Immediate by reflexivity for 0 number of steps.

Proof of Lemma 1 (Inductive case)

- ▶ Suppose $\langle e, \sigma \rangle \longrightarrow \langle e_1, \sigma_1 \rangle \longrightarrow^* \langle n, \sigma' \rangle$.
- ▶ Show $\langle e + e_2, \sigma \rangle \longrightarrow^* \langle n + e_2, \sigma' \rangle$.
- ▶ By induction,
 $\langle e_1 + e_2, \sigma_1 \rangle \longrightarrow^* \langle n + e_2, \sigma'' \rangle$.
- ▶ By LADD, $\langle e + e_2, \sigma \rangle \longrightarrow \langle e_1 + e_2, \sigma_1 \rangle$.
- ▶ Result follows by transitivity:

$$\langle e + e_2, \sigma \rangle \longrightarrow \langle e_1 + e_2, \sigma_1 \rangle \longrightarrow^* \langle n + e_2, \sigma'' \rangle$$

Lemma 2

For all e, e', σ , and n , if $\langle e, \sigma \rangle \longrightarrow \langle e', \sigma'' \rangle$ and $\langle e', \sigma'' \rangle \Downarrow \langle n, \sigma' \rangle$, then $\langle e, \sigma \rangle \Downarrow \langle n, \sigma' \rangle$.

Proof of Lemma 2

(part of homework)

Break: Easier in Small-step or Large-step?

- ▶ Proving totality
- ▶ Dealing with non-termination
- ▶ Expressing interleaving / concurrency
- ▶ Your ideas?

Totality

$$\begin{aligned} &\forall e \in \mathbf{Exp}, \sigma \in \mathbf{Store}, \\ &\exists n \in \mathbf{Int}, \sigma' \in \mathbf{Store}, \\ &\langle e, \sigma \rangle \Downarrow \langle n, \sigma' \rangle \end{aligned}$$

Stuck vs Non-Terminating

$$e ::= \dots$$

- | stop
- | loop

$$\text{LOOP} \frac{}{\langle \text{loop}, \sigma \rangle \longrightarrow \langle \text{loop}, \sigma \rangle}$$
$$\text{LOOP}_{\text{LRG}} \frac{\langle \text{loop}, \sigma \rangle \Downarrow \langle n, \sigma' \rangle}{\langle \text{loop}, \sigma \rangle \Downarrow \langle n, \sigma' \rangle}$$

Rules Add (Original)

$$\text{LADD} \frac{\langle e_1, \sigma \rangle \longrightarrow \langle e'_1, \sigma' \rangle}{\langle e_1 + e_2, \sigma \rangle \longrightarrow \langle e'_1 + e_2, \sigma' \rangle}$$

$$\text{RADD} \frac{\langle e_2, \sigma \rangle \longrightarrow \langle e'_2, \sigma' \rangle}{\langle n + e_2, \sigma \rangle \longrightarrow \langle n + e'_2, \sigma' \rangle}$$

$$\text{ADD} \frac{\text{where } p \text{ is the sum of } n \text{ and } m}{\langle n + m, \sigma \rangle \longrightarrow \langle p, \sigma \rangle}$$

Rules Add (Interleaving)

$$\text{LADD} \frac{\langle e_1, \sigma \rangle \longrightarrow \langle e'_1, \sigma' \rangle}{\langle e_1 + e_2, \sigma \rangle \longrightarrow \langle e'_1 + e_2, \sigma' \rangle}$$

$$\text{RADD} \frac{\langle e_2, \sigma \rangle \longrightarrow \langle e'_2, \sigma' \rangle}{\langle e_1 + e_2, \sigma \rangle \longrightarrow \langle e_1 + e'_2, \sigma' \rangle}$$

$$\text{ADD} \frac{\text{where } p \text{ is the sum of } n \text{ and } m}{\langle n + m, \sigma \rangle \longrightarrow \langle p, \sigma \rangle}$$

Concurrency

$$\text{LCON} \frac{\langle e_1, \sigma \rangle \longrightarrow \langle e'_1, \sigma' \rangle}{\langle e_1 \parallel e_2, \sigma \rangle \longrightarrow \langle e'_1 \parallel e_2, \sigma' \rangle}$$

$$\text{RCON} \frac{\langle e_2, \sigma \rangle \longrightarrow \langle e'_2, \sigma' \rangle}{\langle e_1 \parallel e_2, \sigma \rangle \longrightarrow \langle e_1 \parallel e'_2, \sigma' \rangle}$$

$$\text{CON} \frac{\text{where } p \text{ is the join of } n \text{ and } m}{\langle n \parallel m, \sigma \rangle \longrightarrow \langle p, \sigma \rangle}$$