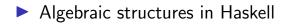
Algebraic Structures CS 152 (Spring 2022)

Harvard University

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# Today, we will learn about

- Type constructors
   Lists, Options
- Algebraic structures
  - Monoids
  - Functors
  - Monads



# Type Constructors

#### A type constructor creates new types from existing types

# Type Constructors

- A type constructor creates new types from existing types
  - E.g., product types, sum types, reference types, function types, ...

Lists

- Assume CBV λ-calc with booleans, fixpoint operator μx:τ. e
- Expressions  $e ::= \cdots | []$   $| e_1 :: e_2 | \text{ isempty}? e | \text{ head } e$  | tail eValues  $v ::= \cdots | [] | v_1 :: v_2$ Types  $\tau ::= \cdots | \tau \text{ list}$ Eval contexts  $E ::= \cdots | E :: e | v :: E$ | isempty? E | head E | tail E

#### List inference rules

isempty? []  $\longrightarrow$  true isempty?  $v_1 :: v_2 \longrightarrow$  false head  $v_1 :: v_2 \longrightarrow v_1$  tail  $v_1 :: v_2 \longrightarrow v_2$  $\Gamma \vdash e_1 : \tau$   $\Gamma \vdash e_2 : \tau$  list  $\Gamma \vdash []: \tau$  list  $\Gamma \vdash e_1 :: e_2 : \tau$  list  $\Gamma \vdash e:\tau$  list  $\Gamma \vdash e:\tau$  list  $\Gamma \vdash e: \tau$  list  $\Gamma \vdash \text{isempty}$ ? e: bool  $\Gamma \vdash \text{head} e: \tau$   $\Gamma \vdash \text{tail} e: \tau$  list

append  $\triangleq \mu f : \tau$  list  $\rightarrow \tau$  list  $\rightarrow \tau$  list.  $\lambda a : \tau$  list.  $\lambda b : \tau$  list. if isompty? *a* then *b* else (head *a*) :: (*f* (tail *a*) *b*)

#### Options

Expressions $e ::= \cdots \mid$  none  $\mid$  some e $\mid$  case  $e_1$  of  $e_2 \mid e_3$ Values $v ::= \cdots \mid$  none  $\mid$  some vTypes $\tau ::= \cdots \mid \tau$  optionEval contexts $E ::= \cdots \mid$  some  $E \mid$  case E of  $e_2 \mid e_3$ 

the type τ option as syntactic sugar for the sum type unit + τ

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- none as syntactic sugar for  $\operatorname{inl}_{\operatorname{unit}+\tau}()$

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- none as syntactic sugar for  $\operatorname{inl}_{\operatorname{unit}+\tau}()$
- some e as syntactic sugar for inr<sub>unit+τ</sub> e

#### Monoids

#### Monoids

A monoid is a set T with a distinguished element called the *unit* (which we will denote *u*) and a single operation *multiply* :  $T \rightarrow T \rightarrow T$  that satisfies the following laws.

$$\begin{aligned} \forall x \in T. \ \textit{multiply} \ x \ u &= x & \text{Left id.} \\ \forall x \in T. \ \textit{multiply} \ u \ x &= x & \text{Right id.} \\ \forall x, y, z \in T. \ \textit{multiply} \ x \ (\textit{multiply} \ y \ z) &= \\ & \textit{multiply} \ (\textit{multiply} \ x \ y) \ z & \text{Assoc.} \end{aligned}$$

### Monoid examples

- Integers with multiplication.
- Integers with addition.
- Strings with concatenation.
- Lists with append.

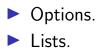
### Functors

#### Functors

A functor associates with each set A a set  $T_A$ ; has a single operation  $map:(A \rightarrow B) \rightarrow T_A \rightarrow T_B$  that takes a function from A to B and an element of  $T_A$  and returns an element of  $T_B$ 

$$orall f \in A 
ightarrow B, g \in B 
ightarrow C.$$
  
 $(map \ f); (map \ g) = map \ (f;g)$  Distributivity  
 $map \ (\lambda a: A. a) = (\lambda a: T_A. a)$  Identity

#### Functor examples



#### Monads

#### Monads

A monad associate each set A with a set  $M_A$ . Two operations:

$$\blacktriangleright$$
 return :  $A \rightarrow M_A$ 

▶ bind : 
$$M_A \rightarrow (A \rightarrow M_B) \rightarrow M_B$$

### Monad laws

$$\begin{aligned} \forall x \in A, f \in A \to M_B. \\ bind (return x) f = f x & \text{Left id.} \\ \forall am \in M_A. bind am return = am & \text{Right id.} \\ \forall am \in M_A, f \in A \to M_B, f \in B \to M_C. \\ bind (bind am f) g = \\ bind am (\lambda a: A. bind (f a) g) & \text{Assoc.} \end{aligned}$$

## **Option monad**

#### return: $\tau \rightarrow \tau$ option bind: $\tau_1$ option $\rightarrow (\tau_1 \rightarrow \tau_2 \text{ option}) \rightarrow \tau_2$ option

## Algebraic structures in Haskell

- https://www.haskell.org/
- Pure functional language
- Call-by-need evaluation (aka lazy evaluation)
- Type classes: mechanism for ad hoc polymorphism
  - Declares common functions that all types within class have
  - We will use them to express algebraic structures in Haskell

# Why Monads?

- Monads are very useful in Haskell
- Haskell is pure: no side effects
- But side effects useful!
- Monadic types cleanly and clearly express side effects computation may have
- Monads force computation into sequence
- Monads as type classes capture underlying structure of computation
  - Reusable readable code that works for any monad

## Further Reading

- Monadic Parsing in Haskell (Functional Pearl) https://www.cs.nott.ac.uk/~pszgmh/pearl.pdf
- Free Monads https://okmij.org/ftp/Computation/ free-monad.html