Harvard School of Engineering and Applied Sciences — CS 152: Programming Languages

Simply-typed lambda calculus (Lecture 11)
Section and Practice Problems

Week 6: Tue Feb 28–Fri Mar 3, 2023

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1 Simply-typed lambda calculus

(a) Add appropriate type annotations to the following expressions, and state the type of the expression.

(i) $\lambda a.\, a + 4$

(ii) $\lambda f.\, 3 + f\ ()$

(iii) $(\lambda x.\, x)\ (\lambda f.\, f\ (f\ 42))$

(b) For each of the following expressions, give a derivation showing that the expression is well typed.

(i) $(\lambda f\!:\!\textbf{int} \rightarrow \textbf{int}.\, f\ 38)\ (\lambda a\!:\!\textbf{int}.\, a + 4)$

(ii) $\lambda g\!:\!(\textbf{int} \rightarrow \textbf{int}) \rightarrow (\textbf{int} \rightarrow \textbf{int}).\, g\ (\lambda c\!:\!\textbf{int}.\, c + 1)\ 7$

(iii) $\lambda f\!:\!\textbf{int} \rightarrow \textbf{int}.\, \lambda g\!:\!\textbf{int} \rightarrow \textbf{int}.\, \lambda x\!:\!\textbf{int}.\, g\ (f\ x)$

2 Type soundness

(a) Recall the substitution lemma that we used in the proof of type soundness.

Lemma (Substitution). *If* $x\!:\!\tau' \vdash e\!:\!\tau$ *and* $\vdash v\!:\!\tau'$ *then* $\vdash e\{v/x\}\!:\!\tau$.

Using the definition of substitution given in Assignment 2, prove this lemma. You may assume that v does not have any free variables (i.e., $FV(v) = \emptyset$).

Remember to state what set you are performing induction on and what the property is that you are proving for every element in that set. If you are not sure what cases you need to consider, or what you are able to assume in each case of the inductive proof, we strongly suggest that you write down the inductive reasoning principle for the inductively defined set.

(b) Recall the context lemma that we used in the proof of type soundness.

Lemma (Context). *If* $\vdash E[e_0]\!:\!\tau$ *and* $\vdash e_0\!:\!\tau'$ *and* $\vdash e_1\!:\!\tau'$ *then* $\vdash E[e_1]\!:\!\tau$.

Prove this lemma.

Remember to state what set you are performing induction on and what the property is that you are proving for every element in that set. If you are not sure what cases you need to consider, or what you are able to assume in each case of the inductive proof, we strongly suggest that you write down the inductive reasoning principle for the inductively defined set.