Uncountable Sets

Countably Infinite

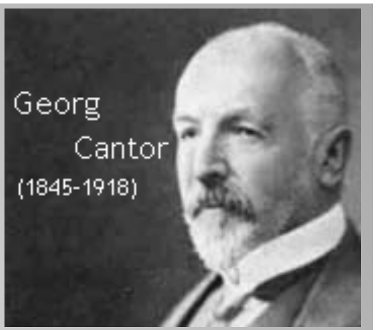
There are as many natural numbers as integers

0 1 2 3 4 5 6 7 8 …

0, –1, 1, –2, 2, –3, 3, –4, 4 …

f(n) = n/2 if n is even, -(n+1)/2 if n is odd
is a bijection from Natural Numbers → Integers

Infinite Sizes



Are all infinite sets the same size?

NO!

Cantor's Theorem

shows how to keep finding bigger infinities.

$P(\mathbb{N})$

- How many sets of natural numbers?
- The same as there are natural numbers?
- Or more?

Countably Infinite Sets

$\{0,1\}^*$ ::= {finite bit strings}

… is countably infinite

Proof: List strings shortest to longest, and alphabetically within strings of the same length

Countably infinite Sets

$\{0,1\}^* = \{e, 0, 1, 00, 01, 10, 11, \ldots\}$

$= \{$ e, $\quad\quad\quad\quad\quad\quad\quad$ $= \{ f(0),$

0, 1, $\quad\quad\quad\quad\quad\quad\quad$ $f(1), f(2),$

00, 01, 10, 11, $\quad\quad$ $f(3), f(4), \ldots\}$

000, … $\}$

Uncountably Infinite Sets

What about infinitely long bit strings? Like infinite decimal fractions but with bits

Claim: $\{0,1\}^{\omega}$ ::= {∞-bit strings} is uncountable.

Diagonal Arguments

Suppose $s_0, s_1, s_2, \ldots \in \{0,1\}^{\omega}$

	0	1	2	3	.	.	.	n	n+1	.	.	.
s_0	0	0	1	0	.	.	.	0	0	.	.	.
s_1	0	1	1	0	.	.	.	0	1	.	.	.
s_2	1	0	0	0	.	.	.	1	0	.	.	.
s_3	1	0	1	1	.	.	.	1	1	.	.	.
.	.	.			1							
.	.		.			1						
	.			.			0					

Diagonal Arguments

- Suppose $s_0, s_1, s_2, \ldots \in \{0,1\}^\omega$

	0	1	2	3	.	.	.	n	n+1	.	.	.
s_0	1	0	1	0	.	.	.	0	0	.	.	.
s_1	0	0	1	0	.	.	.	0	1	.	.	.
s_2	1	0	1	0	.	.	.	1	0	.	.	.
s_3	1	0	1	0	.	.	.	1	1	.	.	.
.	.	.			0							
.	.		.			0						
	.			.			1					

Diagonal Arguments

Suppose $s_0, s_1, s_2, \ldots \in \{0,1\}^{\omega}$

...differs from every row!

So $\{0,1\}^{\omega}$ cannot be listed:

this diagonal sequence

will be missing

Cantor's Theorem

For every set, A (finite or infinite), there is no bijection A↔P(A)

There is no bijection $A \leftrightarrow P(A)$

Pf by contradiction: suppose $f: A \leftrightarrow P(A)$ is a bijection. Let

$W ::= \{a \in A \mid a \notin f(a)\}$, so for any a,

$$a \in W \text{ iff } a \notin f(a).$$

f is a bijection, so $W = f(a_0)$, for some $a_0 \in A$.

$$(\forall a)\ a \in f(a_0) \text{ iff } a \notin f(a).$$

There is no bijection $A \leftrightarrow P(A)$

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$$a \in W \text{ iff } a \notin f(a).$$

f is a bijection, so $W = f(a_0)$, for some $a_0 \in A$.

$$a_0 \in f(a_0) \text{ iff } a_0 \notin f(a_0).$$

contradiction

So P(ℕ) is uncountable

P(ℕ)

= set of subsets of ℕ

↔ $\{0,1\}^\omega$

↔ infinite "binary decimals" representing reals in the range 0..1