1. Multiplying and dividing an integer $n$ by 2 only requires a one digit left or right shift of the binary representation of $n$, which are hardware-supported fast operations on most computers. Here is a state machine, $R$, that computes the product of two nonnegative integers $x$ and $y$ using just these shift operations, along with integer addition:

$$\text{states} := \mathbb{N}^3$$

(triples of nonnegative integers)

$$\text{start state} := (x, y, 0)$$

$$\text{transitions} := (r, s, a) \rightarrow \begin{cases} 
(2r, s/2, a) & \text{for even } s > 0, \\
(2r, (s - 1)/2, a + r) & \text{for odd } s > 0.
\end{cases}$$

(a) Verify that $P((r, s, a)) := rs + a = xy$ is a preserved invariant of $R$.

(b) Prove that $R$ is partially correct: if $R$ reaches a final state—a state from which no transition is possible—then $a = xy$. 

*Credit: Adapted from Albert R. Meyer / MIT 6.042