ES202 Lecture Notes #10

1. The LQ problem

\[
J = \frac{1}{2} \left( x^T S_f x \right) t + \frac{1}{2} \int_t^{t_f} \left[ x^T Q(t) x + x^T N(t) u + u^T N^T(t) x + u^T R(t) u \right] dt
\]

\[
\dot{x} = A(t) x + B(t) u
\]

\[S_f, Q, R, [Q, N, N^T, R] > 0\]

which can be solved both via DP or via calculus of variations.

2. For DP

\[-\frac{\partial V}{\partial t} = \text{Min}_{u} \left[ V_x (A x + B u) + \frac{1}{2} \left[ x^T \begin{bmatrix} Q & N \\ N^T & R \end{bmatrix} x \right] \right]\]

To solve it by separation of variables we postulate that \( V(x,t) = \frac{1}{2} \left( x^T S(t) x \right) \) taking a hint from the terminal condition \( S(t_f) = S_f \). Minimizing the Hamiltonian w.r.t. \( u \), we get

\[u_{\text{opt}} = -R^{-1} \left[ B^T S + N^T \right] x\]

Substituting back into the HJB PDE,

\[- \frac{dS}{dt} = S(A-BR^{-1}N^T) + A^T S + Q - SBR^{-1}B^TS \quad ; \quad S(t_f) = S_f \quad (*)\]

known as the Matrix Riccati Equation.

3. Note that if \( N=0 \) than Eq.\((*)\) simplifies to

\[- \frac{dS}{dt} = SA + A^T S + Q - SBR^{-1}B^T S \quad ; \quad S(t_f) = S_f \quad (*)\]

Thus if we consider \( A = (A-BR^{-1}N^T) \) and \( Q = (Q-NR^{-1}N^T) \) then there is no loss of generality from here on if we only consider the case of \( N=0 \). Thus, henceforth \( u_{\text{opt}} = R^{-1} B^T S x \).

4. Consider the special case of \( Q=0 \) and \( S_f = a I \) with \( a \rightarrow 0 \) or \( S_{f^{-1}} \rightarrow 0 \).

\[S^{-1}(-SA - A^T S + SBR^{-1}B^TS)S^{-1} \rightarrow -\frac{dS^{-1}}{dt}\]

\[
\begin{align*}
SS^{-1} &= I \quad ; \quad SS^{-1} + S(S^{-1}) = 0 \quad ; \quad S^{-1}SS^{-1} = (S^{-1}) \\
&= -S^{-1} A^T - AS^{-1} + BR^{-1}B^T \\
S^{-1}(t_f) &= 0
\end{align*}
\]

We have

\[S^{-1}(t) = \Phi(t,t_f) S_f^{-1}(t_f) \Phi(t,t_f)^T + \int_t^{t_f} \Phi(t,\tau) BR^{-1}B^T \Phi(t,\tau)^T d\tau\]

which is controllability once again.
5. Now for the calculus of variation approach

\[ H = \lambda^T (Ax+Bu) + \frac{1}{2} [x^TQx + u^TRu] \]

\[ H_u = 0 \implies u = -R^{-1}B^T\lambda \]

\[ \dot{\lambda} = -\lambda^TA - x^TQ ; \quad \lambda^T(t_f) = x^T(t_f)S_f \]

We could of course solve the \(dx/dt\) and \(d\lambda/dt\) equations

\[
\begin{bmatrix}
\dot{x} \\
\dot{\lambda}
\end{bmatrix} =
\begin{bmatrix}
A & -BR^{-1}B^T \\
-Q & -A^T
\end{bmatrix}
\begin{bmatrix}
x \\
\lambda
\end{bmatrix} ;
\begin{bmatrix}
x(t_0) = x_0 \\
\lambda^T(t_f) = x^T(t_f)S_f
\end{bmatrix}
\]

However, a quicker way is the anticipate the result and assume \(\lambda(t) = S(t)x(t)\) and derive once again the Riccati equation for \(S(t)\).