1. \[ \frac{dx}{dt} = A(t)x + B(t)u ; \quad x(t_0) = x_0 \] have general solution given by
\[ x(t) = \Phi(t, t_0)x_0 + \int_{t_0}^{t} \Phi(t, \tau)B(\tau)u(\tau)d\tau \]
where \[ \frac{d\Phi(t, \tau)}{dt} = A(t)\Phi(t, \tau) ; \quad \Phi(t, \tau) = I \]
and
\[ \frac{d\Phi^T(t, \tau)}{d\tau} = -A(t)^T\Phi^T(t, \tau) ; \quad \Phi(t, t) = I \]

2. Consider \( \frac{d\phi_i}{dt} = A\phi_i ; \quad \phi_i(t_0) = 1 \) in the ith component and zero everywhere else; \( i=1,2,\ldots,n \). Then, \( \Phi(t, t_0) \) is \( [\phi^1 | \phi^2 | \ldots | \phi^n] \). Furthermore for \( \frac{dx}{dt} - Ax; \; x(t_0) = x_0 \), \( x(t) = \Phi(t, t_0)x_0 \) by superposition.

3. Remember the calculus formula
\[ \frac{d}{dt} \int_{t_0}^{t(h)} g(t, \tau)d\tau = \int_{t_0}^{t(h)} \frac{d}{dt}g(t, \tau)d\tau + g(t, t)\frac{dh}{dt} \]
Apply this to Eq. (*), we get
\[ \frac{dx}{dt} = A\Phi(t, t_0)x_0 + \int_{t_0}^{t} A(t)\Phi(t, \tau)B(\tau)u(\tau)d\tau + \Phi(t, t_0)\Phi(t_0, t)B(t)u(t) \]
\[ = A(t)x(t) + B(t)u(t) \]
Q.E.D.

Plus, heuristic arguments for the correctness of (*) below:
- Break up \( u(t) \) into rectangular pieces with height \( u(\tau) \) and width \( d\tau \). Pretend in the limit these becomes impulses \( u(\tau)\delta(\tau) \).
- The effect of these impulses on \( x \) is that \( x(\tau+)=Bu(\tau) \)
- The effect of a initial condition at \( x(\tau) \) on \( x(t) \) is \( \Phi(t, \tau)Bu(\tau) \)
- Summing up the effect of all \( x(\tau+)=Bu(\tau) \) for all \( t \) we get the inhomogenous part of (*)

4. Consider \( \frac{dX}{dt} = AX \) with \( X(t_0)=C \); \( X(t)=[X^1(t)|\ldots|X^n(t)] \), then by defintion of \( \Phi \)
\( X(t) = \Phi(t, t_0)C \). Also by definition \( \Phi(t_0, t)X(t)=C \). Now, choose \( C=I \), we get
\( \Phi(t_0, t) = \Phi(t, t_0)^{-1} \). Furthermore, differentiating \( \Phi\Phi^{-1}=I \), we have \( \Phi^{-1}+\Phi\Phi^{-1}=0 \).
Thus,
\[
\frac{d\Phi(t,t_0)}{dt} = -\Phi(t,t_0)^{-1}A(t)
\]
\[
\frac{d\Phi(t_0,t)}{dt} = -\Phi(t_0,t)A(t)
\]
\[
\frac{d\Phi(t_0,t)^T}{dt} = -A(t)^T\Phi(t_0,t)^T
\]

which yield (**)

5. Exercise: prove the general result that
\[
\frac{d}{dt} \det \Phi(t,\tau) = Tr(A)\det \Phi(t,\tau)
\]
using a general 2x2 matrix A