ES 272 Assignment #2
Due: March 14th, 2014; 5pm sharp, in the dropbox outside MD 131 (Donhee Ham office)
Instructor: Donhee Ham (copyright ©2014 by D. Ham)

(Problem 1) The $kT/C$ Noise (50pt)

Imagine an arbitrary passive linear network consisting of any number of resistors (with each generating the Nyquist-Johnson thermal noise), capacitors, and inductors in ambient temperature $T$. Any randomly chosen capacitor in this network will store a mean squared noise voltage of $kT/C$, where $C$ is the capacitance of that particular capacitor. This is expected from the equipartition theorem; the capacitor’s mean energy, $\langle CV_C^2/2 \rangle$ ($V_C$: voltage across the capacitor) is written in the quadratic form, thus, must be equal to $kT/2$, i.e., $\langle V_C^2 \rangle = kT/C$. This problem asks you to show this by direct integration with a particular network example of Fig. 1. The lossy elements, $R_1$, $R_2$, and $R_3$ in the network, generate Nyquist-Johnson thermal noise, which are shown as current sources in the figure. Derive the $kT/C$-noise without appealing to the equipartition theorem, but by directly integrating the power spectral density of the voltage noise across the capacitor.

![Figure 1: kT/C noise.](image)

[Suggested reading on noise and random processes in physics]


(Problem 2) $Q$ enhancement technique (50pt)

This problem explores an inductor $Q$ enhancement technique in connection with the transformer circuit of Fig. 2. The primary coil is the inductor whose $Q$ we seek to increase. The primary and secondary coils are identical (the same inductance $L$; the same parasitic resistance $R$). Assume that the coupling between the two coils is perfect, thus, the mutual inductance, $M$, is equal to $L$. We inject an RF energy into the transformer: most of it enters the primary coil, while its small replica (voltage coupling constant, $\alpha \ll 1$), drives the secondary coil after amplification by voltage gain of $G$ and phase shift by $\phi$. The voltage across, and the current flowing into, the primary coil are denoted as $v_1$ and $i_1$, respectively. $v_2$ and $i_2$ are similarly defined for the secondary coil. The relation between $v_1$ and $v_2$ are $v_2 = \alpha G v_1 e^{i\phi}$.
(a) The input impedance, \( Z(\omega) \), shown in the figure can be expressed as \( Z(\omega) \approx v_1(\omega)/i_1(\omega) \) because most RF input energy flows into the primary coil. Show that \( \text{Re}\{Z(\omega)\} = 0 \) if the overall voltage gain \( \alpha G \) and the phase shift \( \phi \) in the path to the secondary coil satisfy

\[
\alpha G = \frac{2Q_0^2 + 1}{2Q_0^2 \cos \phi - Q_0 \sin \phi}
\]

where \( Q_0 \equiv \omega L/R \) is the quality factor of the coils. Since \( Q \) of the primary coil measured from Port 1 is given by \( Q(\omega) = \text{Im}\{Z(\omega)\}/\text{Re}\{Z(\omega)\} \), when \( \alpha G \) and \( \phi \) satisfy (1), the primary coil \( Q \) becomes very large (infinite in theory) with very small (zero in theory) net power dissipation in the primary coil. This effect has been experimentally demonstrated with \( Q \) values up to several thousands.

(b) Although this technique seems in enhancing inductor \( Q \) per se, there is no benefit from the overall energy efficiency point of view. To examine the energetics, prove that, with the arrangement of \( \text{Re}\{Z(\omega)\} = 0 \), the power dissipation in the primary coil \( R \) is exactly compensated by the magnetic energy/power coupled from the secondary coil. This is why you see zero net power dissipation in the primary coil. Also show that with the arrangement of \( \text{Re}\{Z(\omega)\} = 0 \) the power delivered into the secondary coil is the sum of the power dissipation in the secondary coil \( R \) and the aforementioned magnetic energy/power coupled into the primary coil to compensate the dissipation in the primary coil \( R \). In summary, with the arrangement of \( \text{Re}\{Z(\omega)\} = 0 \), while there still exists power dissipation in the primary coil \( R \), this power comes through the secondary coil, making the primary coil appear lossless. There is no magic.

\[ \text{Figure 2:} \ Q \text{ enhancement technique.} \]

\( \text{(Note) As discussed in class, the quality factor, } Q, \text{ is originally defined for a “resonator” as} \]

\[
Q \equiv \omega_0 \frac{\text{Stored energy}}{\text{Power dissipation}}|_{\omega=\omega_0}
\]

where \( \omega_0 \) is the resonance frequency. For a resonator whose resonance frequency is fixed, \( Q \) is not a function of frequency. Since inductors are not resonators, the original \( Q \) definition above cannot be used for inductors. However, the following frequency-dependent \( Q \) definition might be used instead as the quality factor for inductors:

\[
Q(\omega) \equiv \omega \frac{\text{Stored energy}(\omega)}{\text{Power dissipation}(\omega)}
\]
However, it can be easily shown that \(Q(\omega)\) in (3) is on the same order as, but not exactly the same as, 
\[
Q(\omega) = \frac{\text{Im}\{Z(\omega)\}}{\text{Re}\{Z(\omega)\}}
\]  
(4)
where \(Z(\omega)\) is the frequency-dependent input impedance of a given inductor. Often (4) is used instead of (3) as a matter of convention (and definition) for inductors.

(Problem 3) Energetics of Van der Pol oscillators (30pt)

The dynamics of a self-sustained oscillator consisting of a parallel \(LRC\) resonator and active devices may be described by the Van der Pol differential equation:
\[
\ddot{v} + \omega_0^2 v + \omega_r f(v) \dot{v} = 0
\]  
(5)
where \(v\) is the voltage across the \(LC\) tank, \(f(v) < 0\) for small enough \(|v|\), \(f(v) > 0\) for large enough \(|v|\), \(\omega_0 = 1/\sqrt{LC}\), and \(\omega_r = 1/(RC)\). We will assume that the quality factor \(Q = RC\omega_0\) of the resonator is much larger than 1: \(Q \gg 1\).

(a) Show that the time derivative of the instantaneous tank energy, \(E_{\text{tank}}(t) = (1/2) \cdot [Cv^2(t) + Li^2(t)]\), is approximately given by
\[
\frac{dE_{\text{tank}}}{dt} \approx -\frac{\dot{v}^2 f(v)}{R\omega_0^2}
\]  
(6)
As can be seen, when \(|v|\) is small enough, \(f(v) < 0\) and the tank energy increases, and when \(|v|\) is large enough, \(f(v) > 0\), the tank energy decreases.

(b) For \(Q \gg 1\), we can approximate the steady-state voltage output of the oscillator as \(v(t) \approx v_0 \cos \omega_0 t\) (argue why - one sentence should be sufficient), where \(v_0\) is the amplitude. One form of \(f(v)\) that can be used for the Van der Pol oscillator is \(f(v) = av^2 - b\) where \(a > 0\) and \(b > 0\). Show that \(v_0\) is given by \(v_0 = 2\sqrt{b/a}\) noting that in steady state the net tank energy change per period is zero.

(Problem 4) Power spectral density of a noisy oscillator (80pt)

As seen in class, the voltage output of a noisy oscillator can be expressed as
\[
v(t) = v_0 \cos(\omega_0 t + \phi(t))
\]  
(7)
around its fundamental oscillation frequency \(\omega_0 = 2\pi f_0\), where the oscillator phase, \(\phi(t)\), is a random process. In the presence of only white noise which we will assume in this problem, \(\phi(t)\) is a diffusion process (random walk process). The statistical properties of \(\phi(t)\) are given by
\[
\langle \phi(t) \rangle = 0
\]
\[
\langle \phi(t_1)\phi(t_2) \rangle = 2D \min\{t_1, t_2\}
\]  
(8)  
(9)
which are key signatures of any diffusion process subject to white noise, where \(D\) is the phase diffusion constant. \(\phi(t)\) is also a Gaussian process at any given time \(t\).

(a) Express \(\langle v(t) \rangle\) in terms of \(v_0, \omega_0, D,\) and \(t\). (See Hint below.) What is \(\lim_{t \to \infty} \langle v(t) \rangle\)? Why?

(b) Express the autocorrelation of \(v(t)\), *i.e.*, \(\langle v(t_1) v(t_2) \rangle\), in terms of \(v_0, \omega_0, D,\) and \(t\). What is \(\lim_{t \to \infty} \langle v^2(t) \rangle\)?

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1. \(i\) is the current in the inductor. For a high-\(Q\) case, \(i\) can be approximated as \(i \approx C\dot{v}\).
2. The notation, \(\langle \cdot \rangle\), signifies an ensemble average.
(c) Show that the power spectral density of the oscillator, $S_{v}(\omega)$, is given by

$$ S_{v}(\omega) = v_{0}^{2} \cdot \frac{D}{(\Delta \omega)^{2} + D^{2}} $$

where $\Delta \omega \equiv \omega - \omega_{0}$.

(d) The phase diffusion constant, $D$, of an $LC$ oscillator (with parallel $LC$ tank) is roughly approximated as

$$ v_{0}^{2} \cdot D \approx \frac{kT}{C} \cdot \frac{\omega_{0}}{Q} $$

where $Q$ is the quality factor of the $LC$ tank, $k$ is Boltzmann’s constant, and $T$ is the ambient absolute temperature. Using (10), (11), and the definition of phase noise, show that in the $1/f^{2}$ region where $\Delta \omega \gg D$ the phase noise of the $LC$ oscillator is given by

$$ L\{\Delta \omega\} \sim \frac{kT}{P_{diss}} \cdot \left(\frac{\omega_{0}}{Q\Delta \omega}\right)^{2} $$

where $P_{diss}$ is the averaged RF power dissipated in the tank. This is Leeson’s formula very useful in estimating the oscillator phase noise. Note that the oscillator phase noise is inversely proportional to $Q^{2}$ for a given RF power dissipation in the tank.

(e) Calculate the power spectral density, $S_{\phi}(\omega)$, of $\phi(t)$, and show that it is proportional to $D/\omega^{2}$. Make a quantitative comparison of $S_{\phi}(\omega)$ to $S_{v}(\omega)$, and interpret their relation.

(Problem 5) Quadrature generation using $RC$ polyphase filters (50pt)

Quadrature signals (sine and cosine with the same frequency) are useful in a variety of RF signal processing (e.g., vector modulation and demodulation in wireless transceivers). This problem deals with a quadrature generation technique based on $RC$ networks.

(a) $RC$-$CR$ phase splitter: We can generate quadrature signals using an $RC$-$CR$ phase-shift network shown in Fig. 3(a). $V$ is an input RF signal, $V_{1}$ and $V_{2}$ are corresponding outputs. Calculate $V_{2}/V_{1}$ in the frequency domain, and show that $V_{1}$ and $V_{2}$ have a phase difference of $90^\circ$ at all input frequencies, but the amplitudes of $V_{1}$ and $V_{2}$ have the same magnitude only when the frequency of $V$ is given by $\omega = 1/(RC)$.

(b) 1st-order $RC$ polyphase filter: Figure 3(b) shows the 1st-order $RC$ polyphase filter. Two input terminals are driven by differential RF signals ($V$ and $-V$; frequency: $\omega$) while the other two input terminals are tied to ground. Show, by calculating $V_{2}/V_{1}$ in the frequency domain, that $V_{1}$ and $V_{2}$ have a phase difference of $90^\circ$ at all input frequencies, but their amplitudes have the same magnitude only when $\omega = 1/(RC)$. As $\omega$ deviates from $1/(RC)$, the amplitude mismatch between the quadrature signals increases relatively rapidly. You can mitigate this problem and make a broader band quadrature generator by increasing the order of the polyphase filter. See the next problem.

(c) 2nd-order $RC$ polyphase filter: Figure 3(c) shows a 2nd-order $RC$ polyphase filter. Again among the four input terminals, two terminals are driven by differential RF signals ($V$ and $-V$; frequency: $\omega$) while the remaining two terminals are connected to ground. Show, by calculating $V_{2}/V_{1}$ in the frequency domain, that $V_{1}$ and $V_{2}$ have a phase difference of $90^\circ$ at all input frequencies, but their amplitudes have the same magnitude only when $\omega = 1/(RC)$. (Hint: The input signals at the four input terminals, $(V, 0, -V, 0)$, can be decomposed into $(1/2) \times (V, Ve^{j\pi/2}, Ve^{j2\pi/2}, Ve^{j3\pi/2})$ and $(1/2) \times (V, Ve^{-j\pi/2}, Ve^{-j2\pi/2}, Ve^{-j3\pi/2})$.

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3This formula may be interpreted in analogy with Brownian motion. The displacement $x$ of a particle (mass: $m$) suspended in liquid (viscosity: $\gamma$) satisfies $\langle x^{2}(t) \rangle = 2D t$ where $D$ is the diffusion constant. Einstein showed that $D$ is given by $D = (kT/m) \cdot (1/\gamma)$. $kT/C$ and $\omega_{0}/Q$ in (11) correspond to $kT/m$ and $1/\gamma$ in the Brownian motion. In (11), $\nu_{0}^{2}D$ signifies the diffusion rate of the oscillation point on the limit cycle while $D$ is the diffusion rate of the phase (angle).

Figure 3: (a) RC-CR network. (b) 1st-order RC polyphase network. (c) 2nd-order RC polyphase network. (d) 3rd-order RC polyphase network.
In either vector, the relative phase difference between any neighboring components is \( \pm \pi/2 \). You may exploit this constant phase difference and the symmetry of the polyphase filter to simplify your calculation. Can you see that as compared to the 1st-order filter, the amplitude mismatch between \( V_1 \) and \( V_2 \) has less dependence on the frequency offset from \( \omega = 1/(RC) \) in the 2nd-order polyphase filter? Best way to show this is to plot \( |V_2/V_1| \) vs. \( \omega \) around \( \omega = 1/(RC) \) for both the 1st- and 2nd-order polyphase filter.

(d) 3rd-order \( RC \) polyphase filter: By repeating the procedure of (c), we can show that the ratio of the quadrature signals \( (V_1 \text{ and } V_2) \) in the 3rd-order polyphase filter of Fig. 3(d) is given by (You don’t have to derive this)

\[
\frac{V_2}{V_1} = j \cdot \frac{3\omega RC + (\omega RC)^3}{1 + 3(\omega RC)^2}
\]  

(13)

Check again that the two signals have 90° phase shift at all frequencies, and their magnitudes are the same only when \( \omega = 1/(RC) \). Plot \( |V_2/V_1| \) vs. \( \omega \) for the 3rd-order polyphase filter and compare it to the plots of the previous problem (c). You should be able to see that the 3rd-order polyphase filter can generate quadrature signals over a broader band than the 2nd-order polyphase filter.