Elastic Postbuckling Behavior of Stiffened and Barreled Cylindrical Shells

The initial postbuckling behavior of axially stiffened cylinders is studied with a view to assessing the extent to which various effects such as eccentricity, load eccentricity, and barreling influence the imperfection-sensitivity of these structures to buckling. In most cases, these effects result in an increase in the buckling load of the perfect structure, thereby increasing its imperfection-sensitivity as well. In some instances, however, barreling can significantly raise the buckling load of the shell while reducing its imperfection-sensitivity. The analysis, which is based on Koiter's general theory of postbuckling behavior and is made within the context of Kármán-Donnell-type shell theory, takes into account nonlinear prebuckling deformations and different boundary conditions.

Introduction

Recent buckling studies of stiffened cylindrical shells have pointed to a number of potentially useful ways to substantially increase the resistance of these structures to buckling at little or no weight expense. Some of the most interesting of these have been discussed by Almroth and Bushnell [1] and surveyed by Stein [2]. Included are such effects as stiffener eccentricity, load eccentricity, and the location of the stiffeners themselves. It is not uncommon that computed buckling loads are raised by a factor of two or more if these effects are properly taken into account. Tests have already verified that the increase in the buckling load (and thereby the stiffness of the shell) is considerable.

While most studies have taken advantage of these effects and thereby the resulting increase in the buckling load, few theoretical studies to date have been made based on optimal configurations. The purpose of this paper is to examine the extent to which these effects will lead to more efficient structures.

In this paper we examine the extent to which three of these effects, which have each a large influence on the classical buckling load, influence the structure's postbuckling behavior and its imperfection-sensitivity. Previous shell studies seem to indicate that it is not uncommon to find increased sensitivity going along with increased buckling loads. This was established in some instances for outside vs. inside stiffened cylinders and under axial compression in a study which neglected the prebuckling deformations [3]. Here, a consistent approach is followed which includes account of the nonlinear prebuckling deformations. The analysis is based on Koiter's general theory of postbuckling behavior [4-6] and is made within the context of Kármán-Donnell-type shell theory. All calculations are exact although numerical results follow from numerical integration of ordinary differential equations.

A general outline of the analysis is presented in the next section. No attempt has been made to fill in details since many of them are similar to those covered in Badcock's analysis for the case in which the prebuckling deformations can be ignored [7].

Buckling and Postbuckling Analysis

Generalized Kármán-Donnell equations governing the elastic deformation of curved cylindrical shells have been given in References [3] and [8]. These equations are formulated on the basis of nonlinear strain measures of the Donnell-Mushtari-Vlasov type with the aid of the principle of virtual work. Stiffener properties are "measured out" to arrive at effective bending and stretching stiffnesses for the shell-stiffener combination. The governing equations can be reduced to an equation for moment equilibrium and one compatibility equation written in terms of the normal outward deflection of the shell W and a stress function F, namely:

\[ L_p(W) + L_0[F] = F_{xx}W_{xx} + F_{yy}W_{yy} - 2F_{xy}W_{xy} \]  

\[ L_0[F] = W_{xx}^2 - W_{xy}W_{yy} \]  

Nomenclature

- \( a, b \): postbuckling coefficients; see equation (9)
- \( A_{xx}, A_{yy} \): see equation (4)
- \( A_s \): cross-sectional area of stringer
- \( f \): imperfection-sensitivity measure
- \( R_s \): see equation (4)
- \( H = E'/12(1 - \nu^2) \)
- \( D_{xx}, D_{yy}, D_{xy} \): effective bending stiffnesses
- \( d_s \): stringer spacing
- \( E \): Young's modulus
- \( c_r \): distance from stringer centroid to middle surface—positive for an external stiffener
- \( F \): stress function
- \( H \): see Fig. 2
- \( H_{xx}, H_{yy}, H_{xy} \): effective stretching compliances
- \( I_s \): stringer moment of inertia
- \( L \): cylinder length
- \( L_p, L_0, L_0 \): differential operator defined in equation (3)
- \( M_s \): stringer bending moments
- \( N_s \): stringer contribution to resultant membrane stress
- \( \nu \): number of circumferential wavelengths
- \( P \): compressive axial load

(Continued on next page)
where x and y are the axial and circumferential coordinates in the shell middle surface. The linear differential operators are defined by

\[ L_{Dx} = D_{xax} \frac{d}{d x} + 2D_{xex} \frac{d}{d x} + D_{sx} \frac{d}{d x} + D_{sxy} \frac{d}{d x} \]

\[ L_{Dy} = Q_{xax} \frac{d}{d y} + 2Q_{xex} \frac{d}{d y} + Q_{sx} \frac{d}{d y} + \frac{1}{R_a} \frac{d^2}{d y^2} \]

where \( R \) and \( R_a \) are the circumferential and meridional radii of curvature and formulas for the effective bending and stretching stiffnesses were given in Reference [3] and are listed here in Table 1.

In this paper consideration has been restricted to cylinders (and slightly barreled cylinders) with an isotropic skin and axial stiffening. Only three parameters are needed to characterize the stiffening properties if the torsional rigidity of the stringers is ignored. These are the area ratio \( A_s/d_s \), the bending stiffness ratio \( B_s/D_s \), and the eccentricity ratio \( e_s/L \) where \( A_s \) and \( I_s \) are the area and moment of inertia of the stringer, \( d_s \) is the distance between the stringers and \( e_s \) is the distance from the skin middle surface to the stringer centroid, is taken to be positive when the stiffener is on the outer surface of the shell. The resultant membrane stresses in the skin \( N_x \), \( N_y \), and \( N_{xy} \) and the averaged resultant membrane stress in the axial stringers \( N_s \) are related to the stress function and the normal displacement by (see Table 1 for the coefficients)

\[ N_x + N_s = P_{xy}; \quad N_y = P_{y}; \quad N_{xy} = -P_{xy} \]

Fitch [9] and Cohen [10] have given general developments of Koiter's initial postbuckling theory which are directly applicable to the nonlinear shell theory used in this study. Their treatments are extensions of the analysis given in References [11] and [12] to include the effect of nonlinear postbuckling behavior. In the brief outline of the initial postbuckling analysis which follows, the general theory will not be repeated but results from it will be translated directly into the W-F notation of Krenk-Donnell theory. A precise specification of the different boundary conditions is given in the Appendix.

The antisymmetric postbuckling deformation of the perfect shell can be written as

\[ W^* = w^*(x, y) \]
\[ F^t = -\frac{1}{2} \frac{P}{2\pi R} + f^t(x, P) \]  
\[ \text{(Cont.)} \]

where \( P \) is the total compressive load applied to the cylinder. Even though the prebuckling behavior is nonlinear, the two fourth order differential equations governing \( u^t \) and \( f^t \) are linear in these quantities. These two equations can, in turn, be reduced to a single fourth order equation for \( u^t \):

\[ C_4 u^{xxxx} + \left( C_0 + \frac{P}{2\pi R} \right) u^{xxx} + C_6 u^x = \frac{\nu A_{xx} P}{2\pi E I H_t R^2} + \frac{P}{2\pi E R R_x} \]

(\text{6})

Together with an auxiliary equation for \( f^t \):

\[ H_{xx} f^{xx} = Q_{xx} u^{xx} + \frac{1}{R} u^x - \frac{\nu A_{xx} P}{2\pi E I R} \]

(\text{7})

where \( \gamma = \frac{d\zeta}{dx} \) and

\[ C_4 = D_{xx} + \frac{Q_{xx}}{H_t}; \quad C_0 = \frac{2Q_{xx}}{E H_t}; \quad C_6 = \frac{1}{2\pi R H_{xx}} \]

(\text{8})

The classical buckling load of the perfect structure is denoted by \( P_c \) and in all cases examined here is the load at which a nonaxisymmetric bifurcation from the prebuckling state occurs. An asymptotic perturbation expansion of the solution, valid in the neighborhood of this bifurcation point, can be obtained in the form:

\[ \frac{P}{P_c} = 1 + a(\delta/t) + b(\delta/t)^2 + \ldots \]

(\text{9})

\[ W = W^t + \delta W^{(1)} + \delta^2 W^{(2)} + \ldots \]

\[ F = F^t + \delta F^{(1)} + \delta^2 F^{(2)} + \ldots \]

(\text{10})

where \( \delta \) is the amplitude of the buckling mode \( W^{(1)} \). The prebuckling solution is also expanded about the critical load so that:

\[ W^t = W_{0} + (P - P_c) W_{0}^t + \frac{1}{R} (P - P_c) W_{0}^x + \ldots \]

\[ F^t = F_{0} + (P - P_c) F_{0}^t + \frac{1}{R} (P - P_c) F_{0}^x + \ldots \]

(\text{11})

where \( W_{0} \equiv W_0(x, P_c) \); \( W_{0}^x \equiv \frac{\partial W_0}{\partial x} \) and \( W_{0}^x \equiv \frac{\partial W_0}{\partial x} \), etc.

This expansion generates a sequence of linear boundary value problems. The problem for \( W^t \) and \( F^t \) follows directly from equations (5)-(7):

\[ C_4 \omega^{xxxx} + \left( C_0 + \frac{P_c}{2\pi R} \right) \omega^{xxx} + C_6 \omega^x = \frac{\nu A_{xx}}{2\pi E I H_t R^2} + \frac{1}{2\pi R R_x} \omega^{xxx} \]

\[ H_{xx} \omega^{xx} = Q_{xx} \omega^{xx} + \frac{1}{R} \omega^x - \frac{\nu A_{xx} P}{2\pi E I R} \]

(\text{12})

\[ H_{xx} \omega^{xx} = Q_{xx} \omega^{xx} + \frac{1}{R} \omega^x - \frac{\nu A_{xx} P}{2\pi E I R} \]

(\text{13})

The classical buckling problem is then the linear eigenvalue problem governed by the following equations:

\[ L_\delta W^{(1)} + L_\delta f^{(1)} + P_c \frac{W_{xx}^{(1)}}{2\pi R} - f^{(1)xx} W_{xx}^{(1)} = 0 \]

\[ L_\delta f^{(1)} - L_\delta W^{(1)} + W^{(1)} \frac{f^{(1)x}}{2\pi R} = 0 \]

(\text{14})

where \( \delta \) is the buckling amplitude.

Separation of variables can be obtained in the form:

\[ W^{(1)}(x, y) = w^{(1)}(x) \cos(\alpha y/R) \]

\[ f^{(1)}(x, y) = f^{(1)}(x) \cos(\alpha y/R) \]

(\text{15})

The second order boundary value problem is governed by (here we anticipate \( a = 0 \))

\[ L_\delta W^{(2)} + L_\delta f^{(2)} + 2P \frac{W_{xx}^{(2)}}{2\pi R} - 2f^{(2)xx} W_{xx}^{(2)} \]

\[ - w^{(2)xx} f^{(2)x} = - \frac{1}{2} \left( \frac{\partial}{\partial y} \right)^2 (f^{(2)x}(y)) \]

\[ \cos(2\pi y/R) f^{(2)x}(y) - \cos(2\pi y/R) w^{(2)xx} f^{(2)x} \]

(\text{16})

\[ L_\delta f^{(2)} - L_\delta W^{(2)} + \omega^{(2)xx} f^{(2)x} \]

\[ = \frac{1}{2} \left( \frac{\partial}{\partial y} \right)^2 (w^{(2)xx}) \cos(2\pi y/R) f^{(2)x}(y) - \cos(2\pi y/R) w^{(2)xx} f^{(2)x} \]

These equations can be reduced to two systems of ordinary differential equations with the separation:

\[ W^{(2)} = w_{20}(x) u_2(y) \cos(2\pi y/R) \]

\[ F^{(2)} = f_{20}(x) f_2(x) \cos(2\pi y/R) \]

(\text{17})

Solutions of the above boundary value problems are sufficient to yield the most important information about the postbuckling behavior and the associated imperfection-sensitivity. In every case considered in this paper the first postbuckling coefficient, \( a \), in equation (6) is identically zero since this relation is independent of the sign of the buckling displacement. Therefore, the initial relationship between the load and the buckling displacement amplitude hinges on the sign and magnitude of \( b \). If \( b \) is negative the load carrying capacity diminishes following buckling and the shell is imperfection-sensitive, while if \( b \) is positive the structure retains some ability to support increased loads once bifurcation has taken place. Throughout this paper we have consistently identified \( \delta \) with the amplitude of the buckling mode displacement by normalizing the maximum value of \( W^{(1)} \) to be unity. Thus in equation (9), \( \delta/t \) is the ratio of the buckling amplitude to the skin thickness and not the effective thickness of the skin-stiffener combination. The formula for \( b \) is (Fitch [9])

\[ b = \frac{F^{(1)}(x,y) W^{(1)}(x,y) + 2E_{11}^{(1)}(W^{(1)}(x,y) W^{(2)}(x,y))}{P_c F_{11}^{(1)}(W^{(1)}(x,y) W^{(2)}(x,y))} \]

(\text{18})

where the following shorthand notation has been used:

\[ A^*(B, C) = \int_S \left[ A_{x2} B_{2x} C_x + A_{yy} B_y C_y \right] \]

\[ - A_{x2}(B_y C_x + B_{2x} C_y) \right] dS \]

and the integration is over the entire middle surface of the shell.

The initial slope of the generalized load-deflection curve just following bifurcation indicates further information concerning the extent to which buckling can be expected to be gradual or sudden under the two limiting conditions of loading, i.e., prescribed displacement and prescribed generalized displacement. In every example discussed in this paper the generalized displacement is the average end displacement, and for the cylindrical shell examples this can be written as

\[ \Delta = \int_S (U_{xx} - qW_{xx}) dS = \int_S \left( \varepsilon - \frac{1}{2} W_{xx} - qW_{xx} \right) dS \]

where \( U \) is the axial displacement and \( q \) is the loading eccentricity.

A Taylor expansion of \( \Delta \), using (9)-(11), yields

\[ \Delta \]
pression for \( \alpha \) for the barreled cylinders.

Finally, a measure of the imperfection-sensitivity of the structure is most easily obtained by considering the effect of an initial deviation of the shell middle surface from the perfect configuration in the shape of the buckling mode. Thus with an initial imperfection \( \Delta^I = \delta^I \), the maximum support load \( P_s \) is related to the classical buckling load \( P_c \) and the imperfection amplitude \( \delta \) by the asymptotic formula

\[
P_s/P_c = 1 - 3(2)^{-1/2}(-\delta^I)^{1/2} \delta^{1/4} + \ldots
\]

\[
\geq 1 - 1.80(\delta^I)^{1/2} \delta^{1/4} + \ldots
\]

(21)

Here again, the imperfection amplitude has been normalized with respect to the skin thickness \( t \). When the prebuckling state is a purely membrane one, \( \delta = 0 \), but in the present case in which prebuckling deformations cannot be ignored:

\[
\delta = \frac{\delta^I}{\delta^I} \left[ P_c(\Delta^I W_0, \Delta^I W_0^2) + 2D(\Delta^I W_0, \Delta^I W_0^2) \right]^2
\]

\[
\frac{1}{P_c(\Delta^I W_0, \Delta^I W_0^2) + 2D(\Delta^I W_0, \Delta^I W_0^2)} \left[ \Delta^I W_0 + \Delta^I W_0^2 \right]^2
\]

(22)

\( \delta \)

To complete the outline of the analysis sketched above we close this section with a brief discussion of the numerical method employed to evaluate the results given in the remaining sections. In nondimensional form the several boundary value problems and associated boundary conditions are specified by the following nondimensional quantities: \( E/\rho D, A/\rho t, c, \lambda_1, \lambda_2, \ldots, R/\rho t, Z, nL/R, \) and \( t \). The ordinary differential equations for \( \varphi_{ni}, \psi_{ni}, \psi_0, \psi_1, \varphi_0, \varphi_1, \) etc., are reduced to finite difference form and solved by use of a well-known Gaussian elimination scheme due to Poters [13]. The eigenvalue problem for the classical buckling load and mode can be solved in the usual way, in which the lowest eigenvalue associated with integer values of \( n \) is found, and then the lowest of them all is identified with the classical load. Results were obtained by treating \( nL/R \) as a continuous variable and thus it was not necessary to specify the wavelength or radius of the shell. This procedure is consistent with the fact that the critical value of \( nL/R \) turns out to be a fairly large number and for an \( L/R \) of order unity or less it will be fairly large. Therefore, the results can be regarded as exact for any value of \( L/R \) such that the associated value of \( n \) is integer and approximately for other values of \( L/R \) with an error in the buckling load of order 1, \( n \), and an error in \( b \) and \( \alpha \) of order 1/4.

In any case, application of the Donnell-Mushtari-Vlasov strain measures is restricted to shells for which \( n > 5 \), say. In every example presented in this paper, a single buckling mode is associated with the critical eigenvalue and the expansion (10)-(11) was made in anticipation of this fact.

The expressions for \( b \) and \( \alpha \) can easily be reduced to ordinary integrals over the \( x \) coordinate and these integrals were performed with a standard numerical integration scheme. As a check on accuracy, some examples from reference [8], in which the prebuckling deformation was identically zero for and for which closed form solutions were available, were run as special cases of the present procedure. With sixty integration stations over the half length of the shell, accuracy to within about one tenth of one percent could be obtained for the quantities \( b \) and \( \alpha \). Slight loss of accuracy should be expected for the cases in which prebuckling deformations are important. In this paper all results are given as a ratio of 0.3.

Effect of Stringer Eccentricity on Postbuckling Behavior

Two examples have been chosen to illustrate the effect of stringer eccentricity on the buckling load and on the initial postbuckling behavior of an axially stiffened cylinder under axial compression. The buckling load, the imperfection-sensitivity parameter \( b \) and the initial slope \( \theta \) of the load-shortening curve are plotted in Fig. 1 as a function of the length parameter \( \alpha \).

![Table 2](image)

<table>
<thead>
<tr>
<th>( n )</th>
<th>( b )</th>
<th>( \theta )</th>
<th>( \Delta^I )</th>
<th>( P_c )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.6</td>
<td>0.3</td>
<td>0.01</td>
<td>0.05</td>
</tr>
<tr>
<td>2</td>
<td>0.8</td>
<td>0.4</td>
<td>0.02</td>
<td>0.10</td>
</tr>
<tr>
<td>3</td>
<td>1.0</td>
<td>0.5</td>
<td>0.03</td>
<td>0.15</td>
</tr>
</tbody>
</table>

* There is a discrepancy between Fitch's result [9] for this expression and that of Smith [10]. We have rederived the general result using each of the two formal procedures used by these authors and find Fitch's expression to be correct.

| Location of the stringers on the outer surface of the shell clearly enhances the resistance to buckling. Such eccentricity effects were first noted by van der Neut [11] and have been further studied by a number of investigators, most recently by Block [12], 16 and Minnich and Busnich [17] who, as in the present study, account for the prebuckling deformations.

<table>
<thead>
<tr>
<th>(P=Poisson ratio)</th>
<th>(E=Young's modulus)</th>
</tr>
</thead>
<tbody>
<tr>
<td>2(1-\nu)</td>
<td>E(1-\nu)^2</td>
</tr>
</tbody>
</table>

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Judging from the lower two plots in Figs. 1 and 2 the outside-stiffened barreled shell should be the most imperfection-sensitive of the two configurations at least over some of the range of \( L \).

A similar condition was reached in Reference [8] on the basis of an analysis which predicted prebuckling deformations. The shell considered in Fig. 1 is identical to one of the examples analyzed in Reference [3] and a comparison of the two sets of results indicates that prebuckling deformations must be correctly accounted for if accurate quantitative results are to be obtained. However, the trends indicated by the simpler analysis of Reference [3] are correct.
The results shown in Fig. 2 indicate that the imperfection-sensitivity of this shell as measured by $b$ diminishes steadily as $Z$ becomes large. Nevertheless, the initial slope $\theta$ of the load-shortening curve is very negative, which suggests that buckling (or a nearly perfect shell, at least) will not be gradual even under prescribed end displacement even though the sensitivity to imperfections in shape is very low. This example emphasizes that $b$ and $\theta$ must, in general, be regarded as measures of two different characterizations of a structure. Except for the horizontal stiffness of the stringers which has been neglected in this study, the parameters characterizing the shell in Fig. 2 correspond quite closely to some of the test specimens discussed by Card and Jones [17]. The values of $Z$ associated with these specimens were well above 1000, and the buckling behavior of the outside-stiffened cylinders did seem to be consistent with the theoretical prediction of catastrophic buckling under prescribed end displacement coupled with relatively low imperfection-sensitivity.

Effect of Load Eccentricity on Postbuckling Behavior

The simply supported cylinders discussed in conjunction with Fig. 1 were taken to be supported at the skin middle surface with the axial load acting through the skin middle surface as well. Thus the axial load also induces an end moment about the effective centroid of the skin-stiffener combination. This moment, in turn, induces a compressive hoop stress in the inside-stiffened cylinder which tends to lower the shell's buckling load. If instead, the inside-stiffened cylinder were supported at, for example, the centroid of the stringers, then the prebuckling moment would have the opposite effect. Black [15], [16] and Almenroth and Bushnell [1] have demonstrated the importance of this effect by way of a number of examples.

In Fig. 3 we present one example which illustrates the extent to which the load eccentricity influences the initial postbuckling behavior. The inside-stiffened shell marked "loaded at skin centroid" is the same as that considered in Fig. 1. The curves labeled "loaded at stringers centroid" are for the same shell but, as discussed above, simply supported at the centroid of the stringers (see the Appendix for complete specification of the boundary conditions). Loading eccentricity is clearly a very important factor in determining the buckling load particularly in the lower range of $Z$. At the same time, the imperfection-sensitivity and the tendency for catastrophic buckling go up sharply along with the buckling load, at least in this one example.

Effect of Barreling on Postbuckling Behavior

The example we have chosen to illustrate the effect of a small initial axial curvature on the buckling and postbuckling behavior of a clamped, axially stiffened cylindrical shell under axial compression is shown in Fig. 4. The normalized buckling load together with the parameters $\alpha$ and $\theta$ are plotted for one value of the length parameter $Z$ as a function of the parameter which characterizes the amount of barreling.

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Fig. 3. The effect of load eccentricity on the buckling and postbuckling behavior of axially stiffened cylindrical shells which are simply supported at either the centroid of the skin or the centroid of the stringers and subject to axial compression.

$$\sqrt{1 - \frac{L^2}{R^2}} \approx 8 \sqrt{1 - \frac{t}{H}}$$

The second expression above holds when the rise $H$ at the equator is small compared to the length $L$. Even slight barreling, corresponding to a rise at the equator of only several skin thicknesses, has a pronounced effect on the buckling load.

The outside-stiffened shell is a good deal more imperfection-sensitive than the inside-stiffened one in this example when neither is barreled. Any amount of barreling seems to diminish the sensitivity of the outside-stiffened cylinder, while it increases the sensitivity of the inside-stiffened one over the range in which its buckling load rises rapidly with increasing barreling. However, once the barreling is sufficiently large to cause little further increase in the classical load, then even greater barreling seems to have as its main effect a decrease of the imperfection-sensitivity.

Almroth and Bushnell [1] and Stein [2] discussed examples in which barreling has a much larger effect on the classical buckling load of an inside-stiffened cylindrical shell than its outside-stiffened counterpart so that with a slight amount of barreling the inside-stiffened shell actually has a higher buckling load. We have studied a simply supported shell of this type. The postbuckling behavior predicted is similar to that shown in Fig. 4 for the clamped shells, except that the increase in imperfection-sensitivity of the inside-stiffened cylinder is even much more pronounced in the range in which the buckling load rises rapidly with only small increases in barreling.

Concluding Remarks

As a rough rule, it would appear that most mechanisms of the type discussed here for increasing the buckling load of a stiffened cylindrical shell increase to some extent its imperfection-sensitivity as well. A similar tendency has been noted for other shell configurations—see, for example, the survey by Budiansky and Hutchinson [18] and the paper by Koiter [6]. An important exception to this rule seems to occur in some instances when barreling is employed to increase buckling resistance as has been discussed in the previous section. More extensive studies of this aspect of barreling would probably be worthwhile.

In general, stiffening lowers imperfection-sensitivity. Some tests on carefully prepared cylinders with moderately heavy stiffening suggest that loads close to those predicted for the perfect structures can be obtained, although it is not entirely clear what boundary conditions actually have been enforced in a number of these tests [1, 2]. More lightly stiffened cylinders show much greater discrepancies between predictions for the perfect shells and tests [1]. Recently the assertion has been made to the effect that the classical buckling analysis should adequately predict the actual buckling loads of cylindrical shell structures.
with practical levels of stiffening (19), (20). The present results suggest that perhaps such a blanket assertion may be cautious until further studies and tests to determine optimal configurations of stiffening, barreling, etc., have been carried out. This note of caution seems to be further justified since catastrophic buckling has been observed in a number of the tests reported in the recent literature.

APPENDIX

Boundary Conditions

Clamped Boundary Conditions

\[
\begin{align*}
W - W_x &= 0 & z = 0, L \\ V &= 0 & z = 0, L \\ U - U_x &= 0 & z = 0 \\ U_y &= 0 & z = L
\end{align*}
\]

Equations (23) and (24) imply \( \epsilon_y = 0 \) or in terms of \( W \) and \( F \):

\[
(1 - \nu A_{xx} F_{xx} - \nu A_{yy} F_{yy} - \nu B_{xx} W_{xx} = 0
\]

Equations (23), (24), (25) imply \( 2\epsilon_{yy} = 0 \) or in terms of \( W \) and \( F \):

\[
-2(1 + \nu) A_{yy} F_{yy} - (1 - \nu A_y) F_{yy} + \nu B_{yy} W_{yy} = 0
\]

Thus, the clamped boundary conditions, written in terms of \( W \) and \( F \), are equations (23), (25), and (27). The boundary conditions for each of the sequence of boundary value problems is obtained by substitution of the perturbation expansion into these conditions. It should also be mentioned that the requirement that the tangential displacements \( U \) and \( V \) be single-valued in any circuit about the circumference of the shell can be shown to be satisfied for each of the boundary value problems considered.

Simple Support Boundary Conditions

\[
\begin{align*}
W &= 0 & z = 0, L \\ V &= 0 & z = 0, L \\ N_x + N_y = F_{yy} = -P/2\pi R & z = 0, L \\ M_{xx} + M_{yy} + (N_x + N_y)(\epsilon - \gamma) - N_x \epsilon_x &= 0 & z = 0, L
\end{align*}
\]

The load eccentricity \( \gamma \) and the stringer eccentricity \( \epsilon \) are measured from the skin middle surface and are taken to be positive in the outer direction. Equation (29) implies (26). Equation (31) can be written as

\[
(D + EI/d)W_{xx} + F_{yy}(\gamma - \epsilon_v) + c_v(A_{xx} F_{xx} + A_{yy} F_{yy} + B_{xx} W_{xx}) = 0
\]

Thus, equations (26), (29), (30), and (31) are the boundary conditions considered here. Substitution of the perturbation expansion into these equations yields the conditions for each problem; for example,

\[
w^4 = 0
\]

\[
(D + EI/d)w'''' = (q - c_p)P/2\pi R + c_v(A_{xx} F'''' + A_{yy} F'''' + B_{xx} w'''' = 0
\]

and

\[
B^{(1)} W_{xx}^{(1)} = F^{(1)} W_{xx}^{(1)} = F^{(1)} \epsilon_{xx}^{(1)} = 0 & z = 0, L
\]

for \( i = 1, 2 \).

References


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