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SURVEY OF SOME RECENT WORK ON THE MECHANICS OF NECKING

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Abstract

Recent work on the mechanics of necking and localization in metals is reviewed. Several approaches are discussed including bifurcation analyses of diffuse necking modes and shear band modes and nonlinear imperfection growth analyses. Various configurations are considered including slabs, bars and sheets. Emphasis is placed on the influence of constitutive assumptions on predicted behavior.

1. INTRODUCTION

Necking and localization phenomena in metals and other materials have been receiving increased attention from mechanikers in recent years. In part, this stems from a desire to understand how these phenomena fit into the general theory of nonlinear continuum mechanics. In addition, there are a number of issues related to material forming, testing and failure for which a better understanding of the underlying mechanics appears to be essential for further progress. In metals the phenomena fall within the context of finite strain plasticity theory. Familiarity with this class of problems is rapidly increasing. Although there are still many points to be resolved, particularly with respect to constitutive characterization, much can be learned from existing field equations for the incremental deformation of finitely deformed elastic-plastic solids. The development of numerical methods for dealing with finite strain incremental problems has also progressed rapidly, and this in turn has stimulated interest in the basic theory.

In this review, a selection of recent work will be discussed which falls under the general heading of necking and localization in metals. Our primary aim is to convey a sense of the potential for progress which appears possible on several aspects of the subject and, at the same time, to bring out the fact that the major obstacle to further progress is, in most instances, inadequate constitutive descriptions. No attempt has been made to exhaustively review the subject or to provide an extensive bibliography on necking. Instead, we have drawn from a rather limited number of references and we have tried to bring out the common ground in various aspects of the subject. We begin with a discussion of necking and shear band localization in the plane strain deformation of slabs and go on to discuss necking in bars and thin sheets. The significant effect of material strain-rate dependence is discussed in the last section.

Much of the confusion concerned with the generalization of small strain plasticity theory to a finite strain formulation which existed in the past now seems to have been largely cleared up. A number of fundamental issues do remain to be resolved that are strictly large strain issues, but there appears to be fairly wide-spread acceptance of a general framework for finite strain plasticity. This generalization, with its associated variational principles, uniqueness and bifurcation conditions, is largely due to Hill [1-3]. All the work discussed here for time-independent plasticity fits in this framework.

2. BIFURCATION PHENOMENA IN PLANE STRAIN TENSION

The most thorough study of the relation between various possible bifurcation modes and necking behavior in tension has been for the plane strain deformation of a rectangular slab [4-5]. Take the faces of the slab to be parallel to the x_1 and x_2 axes and assume the initial properties to be orthotropic (possibly isotropic) with respect to these axes. The lateral faces perpendicular to the x_2 axis ($x_2 = \pm a_2$) are traction-free and the two end faces ($x_1 = \pm a_1$) are subject to idealized boundary conditions in which the ends undergo a relative separation in the x_1-direction with vanishing shearing traction. The state of uniform in-plane tension is one possible solution for any amount of relative separation of the ends. If it is assumed that the material is incompressible, then the incremental in-plane deformations from this state are governed by only two instantaneous moduli [6]. For an elastic-plastic solid, which will have more than one possible branch to its rate-constitutive relation, the bifurcation mode will be associated with the branch having the "softest" moduli which includes the proportional loading increment. With $\dot{\epsilon}_{ij}$ as the strain-rate, σ_{ij} as the Cauchy stress and $\overset{*}{\sigma}_{ij}$ denoting its Jaumann derivative,

$$\overset{*}{\sigma}_{11} - \overset{*}{\sigma}_{22} = \frac{1}{2} E_t(\dot{\epsilon}_{11} - \dot{\epsilon}_{22}) \tag{1}$$

$$\overset{*}{\sigma}_{12} = 2\mu\dot{\epsilon}_{12} \tag{2}$$

with $\dot{\epsilon}_{11} + \dot{\epsilon}_{22} = 0$. Thus, E_t is the instantaneous tangent modulus relating increments in true stress and natural strain ($\overset{*}{\sigma}_{11} = E_t\dot{\epsilon}_{11}$) for an increment of pure in-plane tension; μ is the instantaneous shear modulus.

If we consider an elongation history in which the two ends of the slab are separated monotonically, the moduli will have their elastic values (which may be stress dependent) prior to yield. Beyond yield, E_t is assumed to decrease monotonically in the uniform state, while the dependence of μ on the stress in the uniform state depends on the particular choice of constitutive law. For a solid with a smooth yield surface, μ is necessarily the instantaneous elastic shear modulus since the plastic strain increment has no shearing component. For a solid which develops a corner on its yield surface, μ will decrease with increasing stress in the uniform state, although at a slower rate than the tangent modulus. The instantaneous moduli of the J_2 deformation theory of plasticity are often used to model the total loading branch with the softest moduli of a yield surface with a corner. In the small strain range, the shearing modulus for the in-plane deformation of the incompressible slab is given by

$$\mu = E_s/3 \tag{3}$$

according to deformation theory for an initially isotropic solid, where E_s is the secant modulus of the equivalent uniaxial stress-strain curve.

As the slab is elongated in its uniform state the maximum load (per unit length in the x_3-direction) is attained at the stress where E_t has been reduced to

$$\sigma \equiv \sigma_{11} = E_t \tag{4}$$

The first possible bifurcation from the uniform state cannot occur below the stress (4) associated with maximum load as long as $2\mu > E_t$ which will almost always be the case in an elastic-plastic solid. A rather complete study of eigenstresses and eigenmodes covering essentially the entire range of material and geometric parameters has been given in Hill and Hutchinson [5]. For a relatively slender slab, with current length a_1 and thickness a_2 such that

$$\gamma = \pi a_2/(2a_1) << 1 \tag{5}$$

the lowest bifurcation stress occurs just beyond maximum load when

$$\sigma/E_t = 1 + \frac{1}{3}\gamma^2 + \frac{7}{45}\gamma^4 + \dots \tag{6}$$

A dependence on E_t/μ appears first in the term of order γ^6. For a solid which is rigid in shear, i.e. $E_t/\mu = 0$, the lowest bifurcation stress for arbitrary γ is given by the smaller of

$$\frac{\sigma}{E_t} = \frac{1}{2} \pm \frac{\gamma}{\sin(2\gamma)} \tag{7}$$

where the plus goes with a symmetric mode (with respect to $x_2 = 0$) and the minus with an anti-symmetric mode. This limiting solution of Cowper and Onat [7] was one of the first diffuse-mode necking bifurcations to be obtained.

The uniform state admits shorter wavelength bifurcation modes of a diffuse nature at stresses above the lowest bifurcation stress. In fact, there is necessarily an infinity of modes, with all possible combinations of symmetry and anti-symmetry with respect to the two planes $x_1 = 0$ and $x_2 = 0$, before the stress exceeds the value given by

$$\frac{\sigma}{E_t} = 1 + \frac{\sigma}{E_t}\left(\frac{2\mu-\sigma}{2\mu+\sigma}\right)^{1/2} \tag{8}$$

This is the stress at which surface instability modes of arbitrarily short wavelengths set in. In the range of deformation in which (8) applies $E_t/\mu << 1$, unless the hardening rate is quite large, and (8) can be replaced by the approximation

$$\sigma \cong \sqrt{2\mu E_t} \tag{9}$$

At a still higher stress, the governing equations cease to be elliptic and become hyperbolic. Then localized shear-band bifurcation modes become possible. The stress separating the elliptic and hyperbolic regimes is

$$\sigma = E_t\left(\frac{4\mu}{E_t} - 1\right)^{1/2} \tag{10}$$

which for $E_t/\mu << 1$ is approximately

$$\sigma \cong 2\sqrt{\mu E_t} \tag{11}$$

The above bifurcation stresses are derived assuming a uniform state of in-plane tension at the instant of bifurcation, while in an actual history the stresses would cease to be uniform beyond the lowest bifurcation point. With realistic end conditions nonuniform stresses develop upon first application of load and symmetric bifurcation would not be expected. As necking progresses the stress field becomes more nonuniform with increasing values of σ_{22} at the center of the neck. Nevertheless, as long as $\sigma_{11} >> \sigma_{22}$ it can be expected that the estimates for the stresses at which the short wavelength modes and, in particular, the localized shear-band modes set in are reasonably accurate. Imposition of hydrostatic tension or pressure does not alter the state of shear-band formation.

An example which illustrates the above sequence of events has been studied experimentally and theoretically by Asaro [8]. Asaro considered an idealized model of a single crystal in which two families of slip planes are symmetrically disposed about the tensile axis. The normal and slip direction of each plane lie in the x_1-x_2 plane, and the slip directions make an angle ϕ with the tensile axis (x_1). Elastic effects are neglected, and it is assumed that there is no cross-hardening between the two systems. The instantaneous hardening coefficient for each system is h so that $\dot{\tau} = h\dot{\gamma}$, where $\dot{\gamma}$ is the shear strain-rate and $\dot{\tau}$ is the rate of increase of resolved shear stress on the crystallographic slip plane. With both systems active, the incremental constitutive relation is precisely of the form (1) and (2) for in-plane deformations. Asaro finds

$$E_t = \frac{2h}{(\sin 2\phi)^2}, \qquad \mu = \frac{h + \sigma \cos 2\phi}{2(\cos 2\phi)^2} \tag{12}$$

In the test [8] the single crystal displayed diffuse necking at first and then gave way to localized shear bands after further straining. For $\phi = 35°$ corresponding to the slip plane orientation in the single crystal when localization set in, the analysis using (10) with (12) indicates that $\sigma \simeq 15h \cong 7E_t$. The angle which the shear band makes with the tensile axis according to the analysis is approximately 40°. From (6), diffuse necking starts when $\sigma \cong E_t$, and thus shear band localization is postponed well beyond the onset of diffuse necking. Asaro argues that the stress level required for shear band formation under these circumstances is consistent with the expected hardening level at that level of straining. From a metallurgical standpoint, these results are significant because the bands form while the systems are still hardening $(h > 0)$, and in this sense the material is still stable at the macroscopic level. For a crystal orientated for single slip, a localized shear band requires an ideally plastic or strain softening state, assuming the classical Schmid law applies. When the Schmid condition is relaxed, for example, allowing for some influence on slip of the shear stress acting on a cross-slip plane, Asaro and Rice [9] have shown that it is then also possible for shear bands to occur with positive hardening of the slip system. Thus, the conclusion from these two studies is contrary to the commonly held view that highly localized straining in a crystal requires work softening or, at least, non-hardening of the slip systems.

3. NECKING OF BARS UNDER UNIAXIAL TENSION

The conventional wisdom based on Considère's one dimensional analysis is that necking of a bar in uniaxial tension starts at the maximum load. One of the perplexing early results in bifurcation analysis is that necking-type bifurcation modes are not possible in a tensile bar of a rigid-plastic solid with a smooth yield surface. More recently, however, it has been shown [10, 11] that allowance for elasticity leads to a necking bifurcation just slightly beyond the point of maximum load if the bar is not unduly stubby, in general accord with the Considère result. Let L and R be the current length and radius of a uniform round bar whose lateral faces are subject to no traction and whose ends are subject to an imposed uniform relative axial separation with no shearing tractions. For a solid with properties which are initially uniform and isotropic or transversely isotropic with respect to the axis of the cylinder the fundamental solution is the state of uniaxial tension. The lowest bifurcation stress has been obtained for the case of an incompressible elastic-plastic solid with transversely isotropic elastic moduli and with a smooth yield surface [11]. The result is not strongly dependent on the elastic anisotropy. For the case in which the elastic moduli are governed by an isotropic relation between the Jaumann-rate of the Cauchy stress and the strain-rate, with elastic shear modulus μ, the lowest bifurcation stress associated with an axisymmetric necking mode is

$$\frac{\sigma}{E_t} = 1 + \frac{\gamma^2}{8} + \frac{\gamma^4}{192}\frac{\mu}{E_t} + O\left(\gamma^4, \gamma^6 \frac{\mu}{E_t}\right) \quad (13)$$

This is an asymptotic formula valid for slender bars for which $\gamma \equiv \pi R/L \ll 1$. Here, E_t is the tangent modulus at bifurcation relating an increment of true tensile stress to an increment of natural strain (i.e., $\dot{\sigma} = E_t\dot{\epsilon}$).

The maximum load condition is again $\sigma = E_t$. Equation (13) suggests that for a solid rigid in shear, i.e. in the absence of any elasticity with $\mu \to \infty$, no bifurcation is possible, which can be demonstrated directly. For slender specimens with $\gamma \ll 1$, bifurcation will be delayed only slightly beyond maximum load according to (13), even though μ/E_t may be as large as 100. The difficulty associated with the limit of a rigid-plastic solid derives from the assumption of a smooth yield surface. If the material is modeled by a deformation theory of plasticity, the strain-rate is not subject to the constraint imposed by the smooth yield surface, and the delay of bifurcation beyond maximum load will be less than that given by (13). Although (13) does not apply precisely to a bar characterized by deformation theory, the replacement in (13) of μ by $E_s/3$ (from (3)) should give a qualitative indication of the difference involved.

Numerical analyses of necking in round bars have been carried out using several methods: Chen [12] used a Kantorovich method, Needleman [13] used finite elements and, most recently, Norris et al. [14] used finite differences. The results of these calculations differ from one another in some of their details but are in general agreement and are conspicuously successful in displaying the full range of the necking process from essentially uniform straining to advanced necking states. It now appears possible to calculate details of the stress and strain distributions in the neck with reasonable accuracy and thereby assess and improve the approximate formulas of Bridgman and Davidenkov and Spiridonova. These have been widely used in the determination of uniaxial stress-strain data at large strains from necked down specimens. Norris et al. [14] have suggested possible corrections to Bridgman's formulas for the particular steel they considered. They have also computed the dependence of the ratio of lateral surface radius to cylindrical radius at the minimum point of the neck as a function of the reduction in cross-sectional area. This is information which is required in the application of Bridgman's formulas and which until now has only been available from direct experimental measurement. Calculations of this type are costly, and more of them will be needed before the role of strain hardening and other factors have been adequately documented. But they hold promise for supplying stress and strain distributions under large strain conditions which are essential to the understanding of the microscopic fracture process. Recent studies which have made use of detailed stress-strain calculations at finite strain for variously designed test specimens have been reported in Argon et al. [15] and Hancock and Mackenzie [16], and others can be expected to follow.

The bifurcation condition for localized shear bands from a fundamental state of uniaxial tension has been obtained by Rice [17] and Needleman and Rice [18]. The assumption of a smooth yield surface effectively excludes the possibility of shear bands under uniaxial tension since the bifurcation stress required for their formation is on the order of the elastic shear modulus. Under plain strain tension, one also concludes from (10) that shear bands would be rather unlikely at the stress levels at which they are observed if μ is identified with the elastic shear modulus, as required by a smooth yield surface, unless E_t falls to an extremely low value. Rice and Needleman consider an alternative yield behavior associated with a cornered yield surface. Instantaneous moduli at bifurcation for a nearly proportional loading increment are modeled by moduli from a deformation theory-type law. This leads to reduced shear moduli as has already been discussed. For example, in the case of plane strain tension where (10) applies, their deformation theory gives $\mu = E_s/3$, where E_s is the secant modulus of the effective true stress-logarithmic strain curve at the current effective stress level. Further comments on the particular deformation theory they use will be made in the next section.

For a pure power law relation between true effective stress and effective logarithmic strain, i.e.

$$\sigma_e = K\epsilon_e^N \tag{14}$$

Needleman and Rice [18] find that the critical strains for shear band formation (major principal component) are

$$\epsilon = [N(1 - N)]^{1/2} \quad \text{(plane strain tension)} \tag{15}$$

$$\epsilon = [(1 + 3N)(1 - N)/3]^{1/2} \quad \left(\begin{array}{c}\text{uniaxial tension}\\ \text{or axisymmetric straining}\end{array}\right) \tag{16}$$

where N is restricted to be less than 1/2 in (15) and 1/3 in (16). Assuming ductility is limited by shear band formation, the ductility ratio obtained from (15) and (16) is (N < 1/3)

$$\frac{\text{ductility in plane strain tension}}{\text{ductility in uniaxial tension}} = \sqrt{\frac{3N}{1 + 3N}} \tag{17}$$

The above results hold for arbitrary superimposed hydrostatic tension or compression since the material is assumed to be incompressible. Thus, (16) applies at the minimum point of a neck in a round tensile bar where axisymmetric straining occurs. Assuming the onset of shear band formation marks the effective end of homogeneous deformation at the local level, and thus the point of material failure, the results (15) and (16) should serve as upper limits on the amount of straining possible for the two states of straining. Any microscopic mechanisms of material failure such as cracking or void growth (also considered in Needleman and Rice [18]) if present would be expected to reduce the strain to failure and, of course, would be sensitive to superimposed hydrostatic tension or compression. A comparison of the results (15) and (16) has been made in Needleman and Rice [18] with experimental data of Clausing [19] for a variety of ductile steels with yield stresses varying between 280 and 1770 MPa. In uniaxial tension (axisymmetric straining), the experimentally determined strain at failure for all the steels tested ranged between .9 and 1.2. For values of N ranging between 0 and 1/3, the strain for shear band formation varies from .58 to .66 according to (16). Thus, ductility is substantially underestimated by shear band analysis in spite of the fact that it is expected to provide an upper bound. There are several possible explanations for the low predictions. Strictly speaking, the bifurcation analysis of the shear bands provides only the strain at the onset of their formation. It is possible, but probably unlikely, that considerable straining of an almost uniform character occurs beyond bifurcation; that is, that localization proceeds very gradually after it starts. It is also possible that the instantaneous moduli used in Rice [17] and Needleman and Rice [18] to model the effect of a corner are not sufficiently "stiff" and thus permit shear bands to form at lower stress levels than they otherwise might. There is some evidence that this may be the case in that a true finite strain deformation theory, to be discussed in the next section, does predict stiffer moduli and delayed shear band formation substantially beyond (16). Another possible explanation of the low strain for shear band formation from (16) is the neglect of material strain-rate sensitivity. The effect of small amounts of strain-rate dependence on the retardation of diffuse necking are substantial, as will be discussed in the last section, but its effect on shear band formation does not appear to have been studied.

Rudnicki and Rice [20] and Rice [17] have broken new ground in studying the influence of other nonclassical plasticity effects such as pressure dependence of yield, on the bifurcation analysis of material failure. Here, too, details of the constitutive assumptions can have a strong influence on the quantitative predictions. Even where these details are not fully understood, the qualitative implications of these studies are significant.

4. NECKING IN THIN SHEETS

Two approaches to the analysis of necking in thin sheets have been used. One is a bifurcation analysis of the onset of a localized band of straining in a uniform sheet. The other assumes an initial geometric or material imperfection and follows its growth as it develops into a neck. Most work has been carried out within the context of the plane stress assumptions. The two approaches have been detailed in Hutchinson and Neale [21] and will be discussed here in turn.

In the bifurcation analysis, a perfect sheet subject to proportional straining is considered so that the principal in-plane strains are related by

$$\epsilon_2 = \rho\epsilon_1 \tag{18}$$

where ϵ_1 is to be regarded as the major component. For a sheet whose initial properties are isotropic, the range of ρ in which sheet failure may be governed by necking is $-1/2 < \rho < 1$, with $\rho = -1/2$ corresponding to uniaxial tension, $\rho = 0$ to a state of in-plane plane strain tension and $\rho = 1$ to equal biaxial stressing in the plane. Bifurcation into a localization band (i.e. necking band) is considered with stress and strain increments varying across the band and vanishing outside it. The incremental behavior is assumed to be governed by the assumptions of plane stress, and therefore the width of the band, or equivalently the minimum characteristic in-plane wavelength of the localization band, must be long compared to the thickness of the sheet for the analysis to apply. In contrast, the three-dimensional shear band bifurcation analyses of the previous sections were carried out under the tacit assumption that the width of the shear band was small compared to any characteristic overall dimension such as thickness. Such shear bands must be considered as signaling the onset of inherent material failure, whereas the necking modes for thin sheets discussed in this section are macroscopic necks and are not material failures as such.

One of the earliest results was that of Hill [22] for a rigid-plastic sheet governed by the common flow laws, based on a smooth yield surface. For example, for a material whose yield surface is described by the J_2 invariant and whose uniaxial behavior is given by the power law relation (14), localization bands occur only for $-1/2 < \rho < 0$ at the strain

$$\epsilon_1 = N/(1 + \rho) \tag{19}$$

The normal to the band makes an angle $\tan^{-1}(\sqrt{-\rho})$ to the axis of principal strain. For $\rho > 0$, no bifurcation is possible in a rigid-plastic solid if a smooth yield surface is invoked. Allowance for elasticity does not essentially change (19), nor does it alter the conclusion that shear bands are effectively excluded for $\rho > 0$.

A considerable body of experimental work on necking in sheet metals under biaxial conditions now exists, e.g. Azrin and Backofen [23], Hecker [24], and Tadros and Mellor [25]. Many factors influence the strain at which necking sets in including, in particular, anisotropy and material strain-rate dependence. The general consensus is that [19] gives a reasonable estimate of the limit strain outside the neck in the range $-1/2 < \rho < 0$ in a sheet which is approximately isotropic at the start of the straining process and which has small strain-rate dependence. On the other hand, necking is almost always observed to occur in ductile sheet metals for $\rho > 0$ at strain levels typically in the range $N < \epsilon_1 < 2N$ depending on the value of ρ and on other factors. Except for a few materials with large $N(\approx .5)$, the minimum limit strain ϵ_1 occurs under conditions of in-plane plane strain, $\rho = 0$.

To get around the difficulty in the range $\rho > 0$, Stören and Rice [26] relaxed the assumption of a smooth yield surface and invoked a solid which develops a corner at the loading point of the yield surface. For a nealy proportional loading increment they modeled the instantaneous moduli by the moduli of a deformation-type plasticity theory -- the same constitutive law mentioned earlier. It was then found that a necking band is predicted for $0 < \rho < 1$. The band forms normal to the direction of principal strain at the strain

$$\epsilon_1 = \frac{3\rho^2 + N(2+\rho)^2}{2(2+\rho)(1+\rho+\rho^2)} \tag{20}$$

assuming that the power law (14) governs the uniaxial behavior. For $\rho = 0$, the flow theory and the deformation theory coincide, and (19) and (20) both give $\epsilon_1 = N$. For $\rho = 1$, (20) gives

$$\epsilon_1 = (1 + 3N)/6 \tag{21}$$

so that $\epsilon_1 > N$ for $N < 1/3$. For $\rho < 0$, the Stören-Rice calculation gives a result somewhat below (19), but not significantly so.

Although it was not emphasized by Stören and Rice, the formula (20) does not apply to the full range of N if their constitutive law is assumed to hold. For example, for $\rho = 1$ material failure in the form of a shear band will occur at a strain below (21) if $N > 1/3$. This shear band cannot be predicted by the plane stress analysis but is the outcome of an analysis such as those described in the previous section. Needleman and Tvergaard [27] have made a detailed study of the various bifurcation modes from a uniform sheet under equal biaxial straining. They consider diffuse plane stress modes and three-dimensional shear band modes.

Hutchinson and Neale [21] considered an alternative finite strain deformation theory to that employed by Stören and Rice [26]. Whereas the constitutive model of Stören and Rice is only a true deformation theory (i.e. nonlinear elastic solid) in the small strain range, that of Hutchinson and Neale is a true deformation theory at finite strains as well. For proportional loading, the two formulations coincide at finite strains and furthermore the two laws reduce to the same small strain limit. The difference in the two laws shows up in that certain of the instantaneous shearing moduli at finite strains are somewhat larger in the true deformation theory than in the corresponding law used in Storen and Rice for the strain states involved. For $\rho < 0$, the results from Hutchinson and Neale lie between those of Hill (19) and Stören and Rice, while for $0 \leq \rho \leq 1$, the analysis of Hutchinson and Neale gives precisely the same result (20) as Stören and Rice. Furthermore, based on the constitutive law of Hutchinson and Neale, material failure does not intercede at strains below (20). In other words, bifurcation into a necking mode is predicted to occur prior to the formation of shear bands at even larger N-values. A brief discussion of this point for the case $\rho = 1$ is given by Needleman and Tvergaard [27].

Marciniak and Kuczynski [28] introduced imperfections to get around the prediction of no necking when $\rho > 0$ in a perfect sheet governed by a flow theory with a smooth yield surface. They assumed plane stress conditions both inside and outside the developing neck and they take the material to satisfy a flow theory using a J_2 yield surface characterization. For an imperfection in the form of a geometric thickness variation or a material property variation which is a function of only the coordinate perpendicular to the infinitely long band, the problem is one dimensional. Furthermore, because of the plane stress assumption, the stress and strain increments at the minimum point of the neck can be directly solved for in terms of the prescribed strain increments outside the band. In this way, they are able to calculate the dependence of the limiting strain outside the neck on the imperfection amplitude. They find that imperfections do lead to finite limit strains for $\rho > 0$, but it is doubtful that the predicted limit strains are sufficiently low to account for test data when realistic levels of imperfections are assumed. A discussion of a number of recent studies using the Marciniak-Kuczynski (M-K) approach is given in Hutchinson and Neale [21] where this same approach is repeated using the true finite strain J_2 deformation theory to characterize the material. The deformation theory results have the virtue that the limit as the imperfection parameter goes to zero gives the bifurcation results discussed previously. Furthermore, the effect of the imperfection on the deformation theory limit strains is a reduction which is roughly the same for all values of ρ.

For the case of in-plane strain, $\rho = 0$, the two plasticity theories give identical predictions. For a material characterized by (14) in simple tension, the asymptotically exact relation between the maximum attainable strain outside the neck and a small thickness imperfection ξ according to the M-K analysis [21] is

$$\epsilon_1/N \cong 1 - \sqrt{(2\xi)/N} \qquad (22)$$

Here ξ is the percentage reduction of the thickness at the minimum point of the band at the start of straining. An extremely small imperfection has a relatively large effect on the necking strain as a consequence of the square root dependence.

It does not yet appear to be possible to use existing experimental data to infer whether one (or perhaps neither) of the constitutive laws is adequate in the analysis of sheet metal necking. Assuming strain-rate effects can be neglected and assuming the material is isotropic, or approximately so, it seems reasonable to expect that the true deformation theory results should supply a lower bound to the limit strains, given realistic imperfection levels, and the J_2 flow theory should supply an upper bound. The difficulty is that these two theories predict widely differing results for $\rho > 0$. It is possible that a smooth yield surface which properly accounts for initial anisotropy may, when used with the M-K analysis, lead to more realistic predictions [29], although a more extended study [30] indicates that small differences in the shape of the yield surface can have large effects on the predictions. A recent study [31] using kinematic hardening in place of isotropic hardening also gives what appears to give results more in line with experimental observations.

One should not lose sight of the fact that the M-K analysis only applies to necks whose variation across the band are long compared to the thickness of the sheet. In the early stages of neck development, the plane stress assumptions of the M-K analysis are reasonably accurate as long as the wavelength characterizing the width of the band, and thus the imperfection, is more than about four times the sheet thickness [32]. Otherwise the M-K analysis overestimates the growth of the neck. In general, other issues aside, the M-K analysis underestimates the maximum attainable strain outside the neck. Since the band is taken to be infinitely long, we can further expect the analysis to underestimate the limit strain due to an initial imperfection with a realistic aspect ratio. No estimates on the effect of the aspect ratio of the initial imperfection are available.

The major uncertainty in the analysis of necking in thin sheets, and also in the analysis of shear band formation, lies in the proper choice of constitutive law. To a certain extent, many of these same issues have surfaced previously in the theory of plastic buckling [33] but the discrepancy between the predictions of the simple flow and deformation theories of plasticity is exacerbated because necking takes place deeper into the plastic range than is usually the case in plastic buckling. Moreover, even among deformation-type theories, there are some significant differences which show up in the finite strain range, as has been discussed. While the deformation theory moduli may perhaps serve as a useful approximation to the moduli of a true flow theory with a corner in carrying out a bifurcation analysis, they most certainly cannot be used to explore post-bifurcation behavior since, then, distinctly non-proportional loading increments and even elastic unloading almost always occur following bifurcation. This is an important consideration since a localization band or a shear band may develop rather gradually following bifurcation and, if such is the case, bifurcation itself will not necessarily provide a good estimate of the failure state. No phenomenological plasticity theory is available which models the effect of a corner and which can be used for exploring both bifurcation and post-bifurcation behavior in situations typical of those encountered in necking or shear band formation (or in plastic buckling).

5. NECKING RETARDATION DUE TO MATERIAL STRAIN-RATE DEPENDENCE

Materials which are significantly strain-rate dependent can undergo extensive straining prior to necking. Rods of glass at high temperatures and superplastic metals can be elongated by as much as 1000% in uniaxial tension. These materials are sometimes described as being nonlinearly viscous with a tensile relation between stress and strain-rate as [34]

$$\sigma = K\dot{\epsilon}^m \tag{23}$$

where m typically lies in the range $.1 < m \leq 1$. It is less well known that very small amounts of strain-rate dependence can also significantly retard the growth of a neck. This obviously has important implications for various metal forming operations, particularly sheet metal forming. Ghosh [35] has collected data from tensile tests on flat strip specimens of a number of sheet metals with small but varying degrees of strain-rate dependence. The considerable additional straining achieved over what would be expected in the absence of any strain-rate dependence correlates consistently with a strain-rate index m defined by the relation

$$\sigma = K\epsilon^N\dot{\epsilon}^m \tag{24}$$

For values of m ranging from 0 to .06 the additional straining increases from 0 to about 40%. In the analysis of many deformation phenomena other than necking, a value of m this small would mean that strain-rate effects could be safely ignored, assuming the strain-rate is not too large, and the material could be taken to be time independent $(m = 0)$.

Equation (24) has been widely used to incorporate the strain-rate influence on the uniaxial stress-strain relation. The elastic strain is not included but usually can be ignored in the range of strains of interest in necking. Equation (24) cannot be expected to give an adequate description of the material under general strain-rate histories. However, over a range of constant strain-rate histories, (24) can be used to provide an approximate fit to experimental data. It is assumed to provide an approximation to the behavior for histories, such as those encountered in many necking tests, where the strain-rate varies slowly.

An elementary analysis brings out the strong strain-rate dependence of necking in a bar of material characterized by (24). Three dimensional aspects of the stress field which develop in an advanced neck are neglected and the stress across each cross-section is taken to be uniaxial and uniform with resultant equal to the instantaneous value of the tensile load carried by the bar. This assumption is analogous to the plane stress assumption invoked in the M-K analysis of sheet necking. It is strictly accurate only when the characteristic wavelength of the neck is long compared to the diameter of the bar. A full three dimensional solution in Hutchinson and Neal [36] for small amplitude, sinusoidal variations of the radius suggests that the wavelength of the non-uniformity should be greater than about three times the diameter for the long-wavelength assumption to be accurate. For shorter wavelengths, as are characteristic of the more advanced stages of necking, the long-wavelength assumption underestimates the strains which can be achieved outside the neck.

The long-wavelength assumption permits one to express the strain at one cross-section in terms of the strain at any other. With ϵ denoting the logarithmic strain at the minimum point of the neck and ϵ_0 the strain in the uniform sections away from the neck, one finds [36]

$$\int_0^{\epsilon} e^{-\tau/m}\tau^{N/m}d\tau = (1-\eta)^{-1/m}\int_0^{\epsilon_0} e^{-\tau/m}\tau^{N/m}d\tau \qquad (25)$$

where η is the initial fractional deficit of cross-sectional area at the minimum point of the neck. The relation (25) between ϵ and ϵ_0 holds for all tensile load histories and, in particular, is independent of the rate of overall straining under which the tensile test is carried out. Thus, necking retardation effects, while they are a consequence of material strain-rate dependence, are independent of the rate at which the test is conducted.

In the limit of a rate-independent material with $m = 0$, the maximum strain which occurs in the uniform section away from the neck is attained at the maximum load point. For small initial imperfections, i.e. $\eta \ll 1$, one obtains from (25) with $m = 0$

$$\epsilon_0^{max} \cong N[1 - \sqrt{2\eta/N}\ ] \qquad (26)$$

and this displays precisely the same strong imperfection-sensitivity as in the in-plane tension case (22). For $\eta = 0$, the Considère result is retrieved. With $m > 0$, ϵ_0 increases monotonically as ϵ increases, and ϵ_0 approaches a finite limit as $\epsilon \to \infty$. An asymptotic result for ϵ_0^{max} can be obtained when $\eta \ll 1$ for small m satisfying $m < 2\eta$ and $m/N \ll 1$. Under these conditions, the necking retardation $\Delta\epsilon_0$, defined as the maximum attainable ϵ_0 minus (26), is

$$\Delta\epsilon_0 \cong \frac{m}{2}\sqrt{\frac{N}{2\eta}}\,\ell n\,(4\pi\eta/m) \qquad (27)$$

Although (27) is limited to a very small range of m, it does bring out the effect of material rate-dependence. For small m, the retardation increases more rapidly than a linear dependence on m according to ($-m\ \ell n\ m$). Increasing the amplitude of the imperfection η decreases the retardation, although for typical values of m and N a tenfold increase in η reduces the retardation by less than a factor of two [36].

For initial imperfection amplitudes ranging from .1% to 1%, the retardation calculated from (25) varies between .4 and .25 for $m = .05$ and $N = .2$, which are typical values for certain sheet metals. At larger values of m, the strain hardening index N is of secondary importance and (25) predicts the extensive straining characteristic of superplasticity.

Necking retardation due to material strain-rate dependence has also been studied in thin sheets using the M-K analysis [37-39], and results similar to those in uniaxial tension have been found over the entire range of biaxial straining. The inclusion of strain-rate dependence does not appear to resolve the difficulties associated with the simple flow theory. Kocks et al. [40] use a more realistic description of rate-dependent material than (24) and carry out a long-wavelength stability analysis of the type first introduced by Hart [41]. This approach is in the spirit of a classical linearized stability analysis. For small initial geometric or material nonuniformities the approach does give accurate estimates of the early stages of neck development. However, it is somewhat doubtful that anything less than a full nonlinear treatment can be used to predict the maximum strain attainable outside the neck and its dependence of the level of the imperfection. This is somewhat reminiscent of the situation in creep buckling where the classical stability approach which was applied to columns proved to be inadequate in predicting times for creep buckling collapse due to the inherent nonlinearity of the phenomena.

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