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Bifurcations at a spherical hole in an infinite elastoplastic medium

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Abstract. The bifurcation problem of an infinite elastoplastic medium surrounding a spherical cavity and subjected to uniform radial tension or compression at infinity is studied. The material is assumed to be incompressible, and its behaviour is modelled by both hypoelastic (flow theory) and hyperelastic (deformation theory) constitutive relations. No bifurcation was found with the flow theory. Surface bifurcation modes were discovered with the deformation theory in both tension and compression. An independent study is also presented of surface bifurcations of a semi-infinite elastoplastic material under equi-biaxial stress. The critical strain for the half-space coincides with the strain at the spherical cavity at the lowest bifurcation.

1. *Introduction.* In this paper we study a basic bifurcation problem of an infinite elastoplastic medium surrounding a spherical cavity subject to uniform radial tension or compression at infinity.

We begin, in Section 2, by considering the primary path of equilibrium which is simply the spherically symmetric mode of expansion or contraction. That path has already been investigated by Durban and Baruch (1) for a particular hypoelastic material whose constitutive law is a finite strain generalization of the J_2 flow theory of plasticity. Here we adopt the incompressible version of that law and recollect the main results for the primary path. We also point out that the same primary path is obtained for an incompressible hyperelastic material whose constitutive relation is a finite strain generalization of the J_2 deformation theory of plasticity.

Next, in Section 3, we specify Hill's (2) criterion for axisymmetric modes of bifurcation from the primary path. A separation of variables is introduced and the whole problem is reduced to a variational equation involving only one unknown function of the radial coordinate.

Numerical solutions of the governing variational equation, obtained by means of a finite element method, are presented in Section 4 for a few representative cases. No points of bifurcation were found with the flow theory over the range of deformation searched. With the deformation theory, on the other hand, bifurcation modes were discovered in both tension and compression. These modes resemble surface bifurcation modes in the sense that they rapidly decay in the radial direction and are highly oscillatory in the circumferential direction. Furthermore, there are many nearly simultaneous modes at almost the same primary state (eigenvalue).

Finally, in Section 5, we give an independent analysis of surface bifurcations of a semi-infinite half-space of elastoplastic material under a uniform state of equi-biaxial tension or compression. The bifurcation condition is expressed in a simple formula that has the same structure as the equivalent formula derived by Hill and Hutchinson (3) and Young (4) for the analogous problem in plane strain. For deformation theory, the critical strain at bifurcation is very close to that obtained in the cavity problem. Results with the flow theory indicate that surface bifurcations take place at enormously high strains.

2. *The primary path.* Consider a spherical cavity embedded in an infinite isotropic, homogeneous elastoplastic medium. Locate the origin at the centre of the cavity, and assume that the medium is subjected to monotonically applied uniform radial tension or compression at infinity. The surface of the cavity is free of traction.

The primary equilibrium path is then simply the spherically symmetric mode of expansion or contraction. This problem has been solved exactly by Durban and Baruch (1) with the particular constitutive relation

$$\overset{\nabla}{\mathbf{s}} = \mathscr{L} \cdot\cdot \mathbf{D}, \tag{2·1}$$

where $\overset{\nabla}{\boldsymbol{\sigma}}$ is the Jaumann rate of the Cauchy stress $\boldsymbol{\sigma}$, $\mathbf{D}$ is the Eulerian strain-rate and $\mathscr{L}$ the tensor of instantaneous moduli. The hypoelastic relation (2·1) used in (1) is a finite strain generalization of the J_2 flow theory of plasticity. Here we proceed with the incompressible version of (2·1), namely

$$\overset{\nabla}{\boldsymbol{\sigma}} = \mathscr{L} \cdot\cdot \mathbf{D}, \tag{2·2}$$

where $\mathbf{s}$ is the stress deviator and $\mathscr{L}$ is now given by (only plastic loading is involved)

$$\mathscr{L} = \frac{2}{3}E\left[\mathbf{I} - \left(1 - \frac{E_t}{E}\right)\mathbf{NN}\right]. \tag{2·3}$$

Here $\mathbf{I}$ is the fourth-order unit tensor whose Cartesian components are

$$I_{ijkl} = \tfrac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) - \tfrac{1}{3}\delta_{ij}\delta_{kl},$$

$\mathbf{N} = \mathbf{s}/(\mathbf{s}\cdot\cdot\mathbf{s})^{\frac{1}{2}}$ is the unit normal tensor to the yield surface, E is Young's modulus of elasticity and E_t is the tangent modulus. The latter is a known function of the effective stress $\sigma_e = (\tfrac{3}{2}\mathbf{s}\cdot\cdot\mathbf{s})^{\frac{1}{2}}$.

We now recapitulate the main results from (1) for the primary path of the incompressible material (2·2)–(2·3). At each point there is radial straining where the stress state is the sum of uniaxial radial plus hydrostatic stressing. In a polar-spherical system of Eulerian (spatial) coordinates (R, θ, ϕ) with the corresponding unit base vectors $(\mathbf{n}_R, \mathbf{n}_\theta, \mathbf{n}_\phi)$, the stress components are given by

$$\Sigma_R = \int_{\Sigma}^{\Sigma_a} f(\Sigma)\,d\Sigma, \quad \Sigma_\theta = \Sigma_\phi = m\Sigma + \int_{\Sigma}^{\Sigma_a} f(\Sigma)\,d\Sigma, \tag{2·4}$$

where with an obvious notation $\Sigma_R = \sigma_R/E$, etc., are the non-dimensional components of $\boldsymbol{\sigma}$, $m = 1$ when the applied load at infinity is radial tension and $m = -1$ when compression, $\Sigma = \sigma_e/E = m(\Sigma_\theta - \Sigma_R)$ is the non-dimensional effective stress and Σ_a its value on the surface of the cavity. With $\epsilon(\Sigma)$ as the logarithmic strain in uniaxial tension at non-dimensional stress Σ, the function $f(\Sigma)$ is given by

$$f(\Sigma) = \frac{\Sigma}{\exp(\frac{3}{2}m\epsilon) - 1}\frac{d\epsilon}{d\Sigma}. \tag{2·5}$$

A common uniaxial relation, which will be used here to generate some numerical examples, is the Ramberg–Osgood expression

$$\epsilon/\epsilon_0 = \sigma/\sigma_0 + \alpha(\sigma/\sigma_0)^n,$$

where n is the hardening index and α is a numerical constant of order unity. The quantity σ_0 can be regarded as an effective yield stress; $\epsilon_0 \equiv \sigma_0/E$ can be thought of as an effective yield strain but note that (σ_0, ϵ_0) does not fall on the stress–strain curve. In terms of the non-dimensional stress the above is

$$\epsilon = \Sigma + \alpha\epsilon_0^{1-n}\Sigma^n. \tag{2·6}$$

In incompressible uniaxial straining ϵ can be referred to as the effective logarithmic strain and $d\Sigma/d\epsilon = E_t/E$.

The load itself – the radial stress at infinity non-dimensionalized with respect to E – is denoted by mP where P is always positive. The effective stress vanishes at infinity and we find from the first of (2·4) that

$$P = m\int_{\Sigma}^{\Sigma_a} f(\Sigma)\,d\Sigma. \tag{2·7}$$

The principal stretches may be written in the form

$$a_R = \frac{dR}{dr} = \exp(-m\epsilon), \quad a_\theta = a_\phi = \frac{R}{r} = \exp(\tfrac{1}{2}m\epsilon), \tag{2·8}$$

where r is the Lagrangian (material) radial coordinate.

Another useful connection is the differential relation

$$\frac{dR}{R} = -m\frac{f(\Sigma)}{2\Sigma}d\Sigma, \tag{2·9}$$

which after integrating and using the second equation of (2·8) gives

$$\begin{aligned}\frac{R}{a} &= \exp\left[+\frac{m}{2}\epsilon_a - \frac{m}{2}\int_{\Sigma}^{\Sigma_a}\frac{f(\Sigma)}{\Sigma}d\Sigma\right] \\ &= \exp[m\epsilon_a/2]\left(\frac{1-\exp[-3m\epsilon_a/2]}{1-\exp[-3m\epsilon/2]}\right)^{\frac{1}{3}},\end{aligned} \tag{2·10}$$

where a is the Lagrangian (undeformed) radius of the cavity and ϵ_a is the value of ϵ at the surface of the cavity.

Relation (2·10) completes the description of the primary path. Regarding Σ_a as a monotonically increasing time-like parameter, we can evaluate with the expressions given here the stress and deformation fields along the primary path. An interesting feature about that path – at least for the representation (2·6) – is the existence of an asymptotic value for P in tension, P_0, which is given by (2·7) with $m = 1$ and $\Sigma_a \to \infty$. By contrast, a thick-walled sphere reaches a limit point in P at finite Σ_a (5).

With R_a denoting the radius of the deformed cavity in the primary solution, the relation between mP/P_0 and $(R_a - a)/a$ is shown in Fig. 1 for two values of n and for values of α and $\epsilon_0 = \sigma_0/E$ which are representative of common polycrystalline metals.

The primary path just described is a proportional path in the sense that the unit normal tensor to the yield surface $\mathbf{N}$ is kept constant along that path. An identical path is obtained if the hypoelastic relation (2·2)–(2·3) is replaced by the isotropic, incompressible hyperelastic relation

$$\mathbf{s} = \tfrac{2}{3} E_s \mathbf{E}_L, \tag{2·11}$$

where E_s is the secant modulus defined by

$$E_s = \sigma_e/\epsilon = E\,\Sigma/\epsilon \tag{2·12}$$

and the uniaxial relation between σ_e and ϵ is the same as before. The tensor $\mathbf{E}_L$ in (2·11) is the Eulerian logarithmic strain tensor whose principal components, when decomposed on the Eulerian triad, are $\ln a_i$ $(i = 1, 2, 3)$ where a_i are the principal stretches. It is easily verified that (2.11) is derived from the strain energy function

$$W = \frac{1}{3}\int E_s(\Pi)\, d\Pi = \int \Sigma\, d\epsilon, \tag{2·13}$$

where

$$\Pi = \mathbf{E}_L \cdot\cdot\, \mathbf{E}_L = \ln^2 a_1 + \ln^2 a_2 + \ln^2 a_3 = \tfrac{3}{2}\epsilon^2. \tag{2·14}$$

Note that the secant modulus in (2·13) is regarded as a known function of Π or equivalently of Σ. The hyperelastic constitutive relation (2·11) is a finite strain generalization of the J_2 deformation theory of plasticity which was introduced in (6).

By taking the Jaumann rate of (2·11) we obtain its rate form

$$\overset{\nabla}{\mathbf{s}} = \frac{2}{3} E_s\left[\mathbf{I} - \left(1 - \frac{E_t}{E_s}\right)\mathbf{NN}\right] \cdot\cdot\, \overset{\nabla}{\mathbf{E}}_L, \tag{2·15}$$

where $\overset{\nabla}{\mathbf{E}}_L$ is the Jaumann rate of $\mathbf{E}_L$ and $\mathbf{N}$ is again $\mathbf{s}/(\mathbf{s}\cdot\cdot\,\mathbf{s})^{\frac{1}{2}}$. The above can be reduced to the form (2·2) involving the Eulerian strain-rate D, i.e.

$$\overset{\nabla}{\mathbf{s}} = \mathscr{L} \cdot\cdot\, \mathbf{D}, \tag{2·16}$$

where

$$\mathscr{L} = \tfrac{2}{3} E_s\left[\mathbf{I} - \left(1 - \frac{E_t}{E_s}\right)\mathbf{NN}\right] + \mathbf{Q}. \tag{2·17}$$

In principal stress axes the components of $\mathbf{Q}$ can be determined from Hill's (7) formulas for the instantaneous shearing moduli of isotropic elastic solids. In principal axes all components of $\mathbf{Q}$ vanish except the shearing components Q_{1212}, Q_{1313} and Q_{2323} and the associated components from the indicial symmetries $Q_{ijkl} = Q_{jikl} = Q_{klij}$. With E_i as the principal logarithmic strains

$$Q_{1212} = \tfrac{1}{3}E_s[(E_1 - E_2)\coth(E_1 - E_2) - 1] \tag{2·18}$$

and analogous expressions hold for Q_{1313} and Q_{2323}.

In passing we mention a third elastoplastic constitutive relation suggested by Stören and Rice (8). That relation is again a hypoelastic relation like (2·2) but the tensor of instantaneous moduli is obtained from (2·15) with $\overset{\nabla}{\mathbf{E}}_L$ replaced by $\mathbf{D}$, namely

$$\mathscr{L} = \tfrac{2}{3}E_s\left[\mathbf{I} - \left(1 - \frac{E_t}{E_s}\right)\mathbf{NN}\right]. \tag{2·19}$$

The primary path of this material is identical with that of materials (2·3) and (2·16). Similarly, it is possible to construct more constitutive relations that will deform along the same primary path. In the present study however we restrict ourselves mainly to materials (2·3) and (2·17)–(2·18). The third material (2·19) will be considered only in the last part of the paper wherein the half-space problem is discussed.

3. *Bifurcation analysis.* Hill's (2) criterion for bifurcation of an elastoplastic material under increasing load states, for the present problem, that

$$\delta \int_V U\, dV = 0 \tag{3·1}$$

where for an incompressible material

$$U = \tfrac{1}{2}(\overset{\nabla}{\mathbf{s}} - \boldsymbol{\sigma}\cdot\mathbf{D} - \mathbf{w}\cdot\boldsymbol{\sigma})\cdot\cdot\mathbf{L}; \tag{3·2}$$

here $\mathbf{L}$ is the left velocity gradient and $\mathbf{w}$ the spin tensor. The integration in (3·1) is carried throughout the whole volume V. It is understood that all rate quantities in (3·2) represent eigen-quantities which are the difference between the two possible solution-rates. Substituting the constitutive relation (2·2) or (2·16) into (3·2) gives the more convenient form

$$2U = \mathbf{D}\cdot\cdot\mathscr{L}\cdot\cdot\mathbf{D} - (\boldsymbol{\sigma}\cdot\mathbf{D} + \mathbf{w}\cdot\boldsymbol{\sigma})\cdot\cdot\mathbf{L}. \tag{3·3}$$

Considering only axisymmetric modes of bifurcation from the primary path, we can write the eigenvelocity as

$$\mathbf{v} = v_R\mathbf{n}_R + v_\theta\mathbf{n}_\theta, \tag{3·4}$$

where v_R, v_θ are functions of R, θ only. The left gradient of $\mathbf{v}$ is then

$$\mathbf{L} = \nabla\mathbf{v} = \epsilon_R\mathbf{N}_R + \epsilon_\theta\mathbf{N}_\theta + \epsilon_\phi\mathbf{N}_\phi + \gamma_\phi\mathbf{S}_\phi + \omega_\phi\mathbf{R}_\phi, \tag{3·5}$$

where ϵ_R, ϵ_θ, ϵ_ϕ, γ_ϕ are the components of the Eulerian strain rate, ω_ϕ is the only non-vanishing component of the spin tensor and

$$\left.\begin{aligned} \mathbf{N}_R &= \mathbf{n}_R\mathbf{n}_R, \quad \mathbf{N}_\theta = \mathbf{n}_\theta\mathbf{n}_\theta, \quad \mathbf{N}_\phi = \mathbf{n}_\phi\mathbf{n}_\phi,\\ \mathbf{S}_\phi &= \mathbf{n}_R\mathbf{n}_\theta + \mathbf{n}_\theta\mathbf{n}_R, \quad \mathbf{R}_\phi = \mathbf{n}_R\mathbf{n}_\theta - \mathbf{n}_\theta\mathbf{n}_R. \end{aligned}\right\} \tag{3·6}$$

Thus,

$$\mathbf{D} = \epsilon_R\mathbf{N}_R + \epsilon_\theta\mathbf{N}_\theta + \epsilon_\phi\mathbf{N}_\phi + \gamma_\phi\mathbf{S}_\phi, \quad \mathbf{w} = \omega_\phi\mathbf{R}_\phi, \tag{3·7}$$

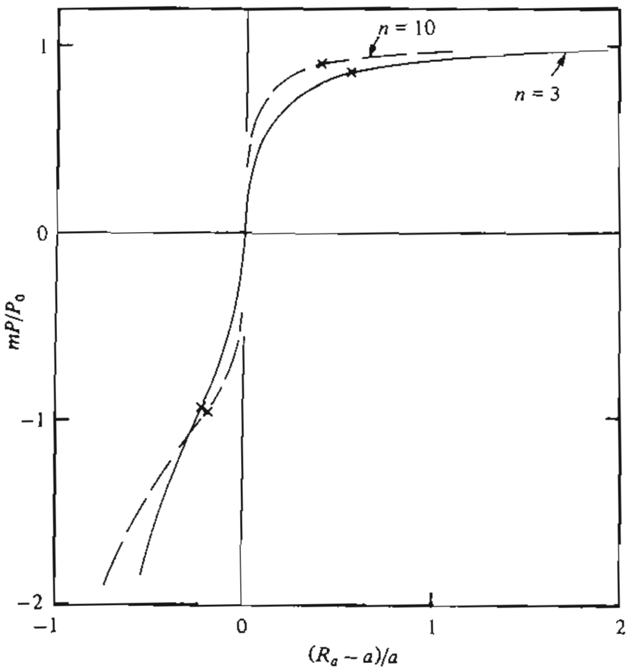

Fig. 1. Relation between normalized radial tension or compression at infinity and growth of hole in primary solution ($\alpha = \frac{3}{7}$, $\epsilon_0 = 0{\cdot}003$). Crosses denote points of first bifurcation.

with

$$\left.\begin{aligned}\epsilon_R = v_R', \quad \epsilon_\theta = \frac{1}{R}(v_R + \dot{v}_\theta), \quad \epsilon_\phi = \frac{1}{R}(v_R + v_\theta \cot\theta),\\ \gamma_\phi = \tfrac{1}{2}\left[v_\theta' + \frac{1}{R}(\dot{v}_R - v_\theta)\right], \quad \omega_\phi = \left[v_\theta' - \frac{1}{R}(\dot{v}_R - v_\theta)\right]\end{aligned}\right\} \tag{3·8}$$

and

$$(\;)' = \frac{\partial(\;)}{\partial R}, \quad (\dot{\;}) = \frac{\partial(\;)}{\partial\theta}. \tag{3·9}$$

The condition of incompressibility $\epsilon_R + \epsilon_\theta + \epsilon_\phi = 0$ can be stated as

$$\frac{1}{R}(R^2 v_R)' + \dot{v}_\theta + v_\theta \cot\theta = 0. \tag{3·10}$$

This equation is satisfied with any pair of the form

$$\left.\begin{aligned}v_R &= \alpha_k \Phi P_k(\cos\theta),\\ v_\theta &= \frac{1}{R}(R^2\Phi)' \dot{P}_k(\cos\theta),\\ \alpha_k &= k(k+1) \quad (k = 1, 2, 3, \ldots),\end{aligned}\right\} \tag{3·11}$$

where Φ is an unknown function of R and $P_k(\cos\theta)$ are the Legendre polynomials. It can be shown that for each k (3·11) results in an exact separation of the differential equations and boundary conditions associated with the variational statement (3·1).

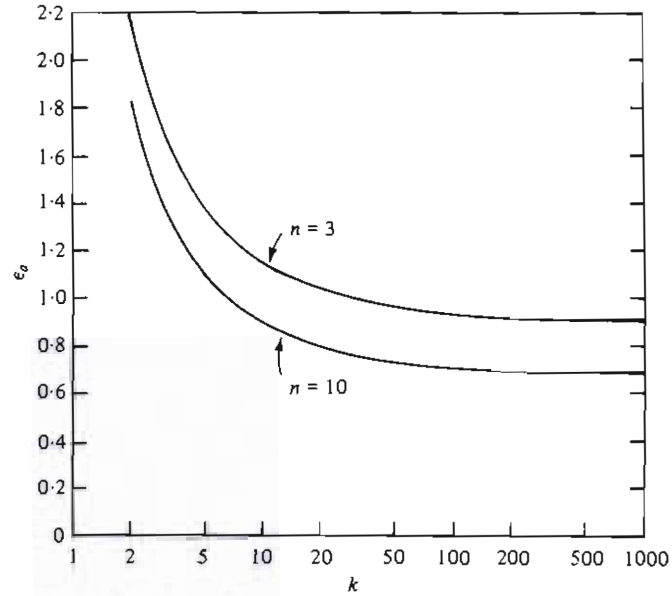

Fig. 2. Effective strain at surface of cavity at occurrence of first bifurcation mode of degree k. Radial tension at infinity with $\alpha = \frac{3}{7}$ and $\epsilon_0 = 0{\cdot}003$.

Substituting (3·5)–(3·7) into (3·3) and observing that for the primary path

$$\boldsymbol{\sigma} = \sigma_R \mathbf{N}_R + \sigma_\theta(\mathbf{N}_\theta + \mathbf{N}_\phi) \tag{3·12}$$

we find that

$$2U/E = (\nu_1 - \Sigma_R)\,\epsilon_R^2 + (\nu_2 - \Sigma_\theta)\,(\epsilon_\theta^2 + \epsilon_\phi^2) + 2(\nu_3 - \Sigma_R - \Sigma_\theta)\,\gamma_\phi^2 + \Sigma_R(\gamma_\phi + \omega_\phi)^2 + \Sigma_\theta(\gamma_\phi - \omega_\phi)^2, \tag{3·13}$$

where $\nu_i = \mu_i/E$ $(i = 1, 2, 3)$ are the non-dimensional instantaneous moduli defined via the quadratic form

$$\mathbf{D}\cdot\cdot\mathscr{L}\cdot\cdot\mathbf{D} = \mu_1 \epsilon_R^2 + \mu_2(\epsilon_\theta^2 + \epsilon_\phi^2) + 2\mu_3 \gamma_\phi^2. \tag{3·14}$$

Thus, for the hypoelastic material (2·3),

$$\nu_1 = \frac{E_t}{E} - \tfrac{1}{3}, \quad \nu_2 = \nu_3 = \tfrac{2}{3} \tag{3·15}$$

In deriving (3·15) we have used the obvious expression

$$\mathbf{NN} = \tfrac{3}{2}(\mathbf{N}_R - \tfrac{1}{3}\boldsymbol{\delta})\,(\mathbf{N}_R - \tfrac{1}{3}\boldsymbol{\delta}), \tag{3·16}$$

where $\boldsymbol{\delta}$ is the second order unit tensor with Cartesian components δ_{ij}.

Similarly, for the hyperelastic material, from (2·16)–(2·18),

$$\nu_1 = \frac{E_t}{E} - \frac{1}{3}\frac{E_s}{E}, \quad \nu_2 = \frac{2}{3}\frac{E_s}{E}, \quad \nu_3 = \frac{E_s}{E}\,\epsilon \coth(\tfrac{3}{2}\epsilon) = \Sigma \coth(\tfrac{3}{2}\epsilon), \tag{3·17}$$

where in the last relation we have used the incompressibility condition $a_R a_\theta^2 = 1$ as well as expressions (2·8) for the principal stretches.

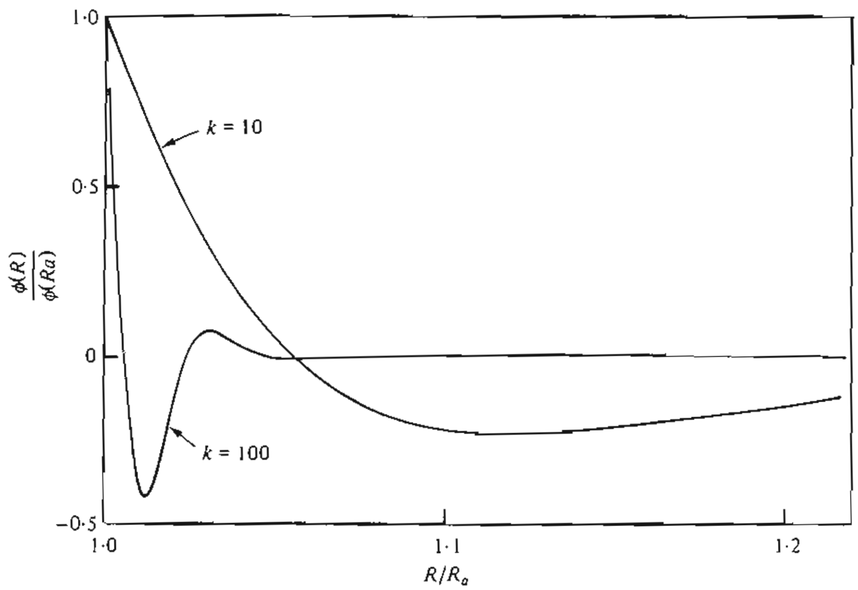

Fig. 3. Radial variation of bifurcation modes of degree $k = 10$ and $k = 100$ for radial tension at infinity with $\alpha = \frac{3}{7}$, $\epsilon_0 = 0{\cdot}003$ and $n = 3$.

As for the third material (2·19), it is observed at once that ν_1, ν_2 are identical with those of (3·17) while ν_3 is equal to ν_2.

Substituting now relations (3·8) into (3·13), using the expression for the velocities (3·11) and separating the result into the R and θ directions, we find that the variational statement (3·1) is reduced to

$$\delta \int_{R_a}^{\infty} \bar{U}\, dR = 0 \tag{3·18}$$

where R_a is the deformed radius of the cavity and $\bar{U}$ is given by

$$\bar{U} = \tfrac{1}{2}\kappa_1(R^2\Phi'')^2 + \tfrac{1}{2}\kappa_2(R\Phi')^2 + \tfrac{1}{2}\kappa_3\Phi^2 + \kappa_4 R^3\Phi''\Phi' + \kappa_5 R\Phi'\Phi + \kappa_6 R^2\Phi''\Phi \tag{3·19}$$

with

$$\left.\begin{aligned}
\kappa_1 &= \nu_3 + \Sigma_R - \Sigma_\theta,\\
\kappa_2 &= 2[\alpha_k(\nu_1 + \nu_2 - \Sigma_R - \Sigma_\theta) + 2\nu_3 - \nu_2 + 7\Sigma_R],\\
\kappa_3 &= (\alpha_k - 2)^2(\nu_3 - \Sigma_R + \Sigma_\theta) + 4(\alpha_k - 2)(\nu_2 - \sigma_\theta),\\
\kappa_4 &= 2(\nu_3 + 2\Sigma_R - \Sigma_\theta)\\
\kappa_5 &= 2(\alpha_k - 2)(\nu_2 + \nu_3 - \Sigma_R - 3\Sigma_\theta),\\
\kappa_6 &= (\alpha_k - 2)(\nu_3 - \Sigma_R - \Sigma_\theta).
\end{aligned}\right\} \tag{3·20}$$

It is now possible to derive from (3·18) the fourth order differential equation for the function Φ along with the corresponding homogeneous boundary conditions. Furthermore, with the aid of relations (2·9)–(2·10) we can eliminate the Eulerian coordinate R and use only the effective stress Σ as the independent variable.

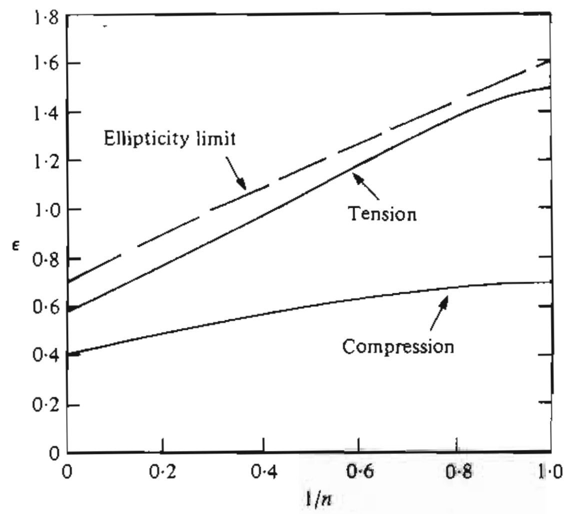

Fig. 4. Effective strain at bifurcation into surface modes for half-space under equal biaxial stressing ($\alpha = \frac{3}{7}$, $\epsilon_0 = 0{\cdot}003$). Essentially identical results are obtained for pure power-law material $\epsilon = K\sigma_e^n$.

Unfortunately, however, attempts to solve the resulting equation analytically were not fruitful.

Exceptionally, for the hyperelastic material (2·16)–(2·18) a further analytical reduction is possible for the bifurcation modes associated with $k = 1$. This has been observed by Haughton and Ogden (9) in their study of bifurcation of incompressible hyperelastic spherical shells under internal pressure.

With $k = 1$ the coefficients κ_3, κ_5, κ_6 from (3·20) vanish and the criterion (3·18) generates then the second-order equation

$$\kappa_1 R^4 \Psi'' + (\kappa_1 R^4)' \Psi' + [(\kappa_4 R^3)' - \kappa_2 R^2]\Psi = 0, \tag{3·21}$$

where $\Psi = \Phi'$. The corresponding boundary conditions are

$$\left.\begin{aligned} &\kappa_1 R\Psi' + \kappa_4 \Psi = 0 \quad \text{at} \quad R = R_a, \\ &\Psi = 0 \qquad\qquad\quad\ \ \text{at infinity} \end{aligned}\right\} \tag{3·22}$$

Inserting coefficients κ_1, κ_2, κ_4 into (3·21), using expressions (3·17) for the instantaneous moduli, arranging and integrating gives the solution of (3·21) in the form

$$\Psi = c_1 \frac{a_\theta}{R^4} \int_{R_a}^{R} \frac{R^4\, dR}{a_\theta^2 \kappa_1} + c_2 \frac{a_\theta}{R^4}, \tag{3·23}$$

where c_1, c_2 are integration constants. Turning now to the boundary conditions we find that the second of (3·22) is satisfied only with $c_1 = 0$ since the first term of the solution (3·23) is unbounded at infinity. (As $R \to \infty$, (2·8) implies that $a_\theta \to 1$, while from ($3{\cdot}20_1$) and ($3{\cdot}17_3$) it follows that $\kappa_1 \to \frac{2}{3}$. Thus, as R becomes large, the first term of (3·23) behaves essentially like $\frac{3}{10} c_1 R$ and the value of c_1 must therefore be taken as zero.) The condition on the surface of the cavity ($3{\cdot}22_1$) can be rewritten as

$R\Psi' + 2\Psi = 0$ at $R = R_a$. Substitution of $\Psi = c_2 a_\theta / R^4$ into that condition and use of (2·5) and (2·9) shows that no nontrivial solution is possible.

Thus the problem for the hyperelastic material does not admit any bifurcation mode with $k = 1$ either in tension or compression.

4. *Numerical results*. The variational equation (3·18) has been discretized for numerical work using a finite element method. The effective stress Σ was used as the independent variable so that the infinite range of integration $[R_a, \infty]$ is transformed to the finite range $[\Sigma_a, 0]$. That range was divided into K sub-intervals with Φ_i and $(d\Phi/d\Sigma)_i$ being the unknowns at the ith node. Cubic interpolation formulas were used for piecewise integration across each element. Equation (3·18) is then replaced by a system of linear homogeneous algebraic equations for the unknowns Φ_i and $(d\Phi/d\Sigma)_i$ with i ranging from 1 to $(K+1)$. The bifurcation criterion requires a non-trivial solution of that system, i.e. the vanishing of the determinant of the coefficients of the algebraic equations. The lowest load at which the determinant vanishes – for a specific k labelling the mode in (3·11) – is the lowest bifurcation load (eigenvalue) of a mode of that order. The range of k treated was from 1 to 1000. At each k the procedure was repeated with a higher number of elements until convergence with sufficient accuracy was reached. The highest number of elements used during the calculations was 240 but in general no more than 120 elements were needed. The behaviour of the eigenmodes on the boundaries – zero velocity at infinity and zero traction rate at the cavity – was checked and verified a posteriori.

The main findings, which are elaborated on below, are the following. No bifurcation occurs when the material is characterized by the flow theory (2·3), except possibly after exceedingly large expansions or contractions of the cavity. Bifurcations occur in tension and compression when the material is characterized by the deformation theory (2·16)–(2·18). Many surface-like modes are clustered just above the lowest bifurcation load which is attained in the limit $k \to \infty$. We will discuss the deformation theory solutions first and defer further discussion of the flow theory until the next section.

A typical dependence of the lowest eigenvalue at each k is plotted against k in Fig. 2. It is most convenient for our discussion to represent the eigenvalues by the associated critical values of the effective strain at the cavity surface ϵ_a. The results shown are for the uniaxial relation (2·6) with $\alpha = \frac{3}{7}$ and $\epsilon_0 = 0{\cdot}003$ and with $n = 3$ and 10 corresponding to the same choices as in Fig. 1. The results in Fig. 2 are for the tensile case ($m = 1$). Of course, the curves in Fig. 2 have a meaning only for integral values of k. No bifurcation was found for $k = 1$ as already noted. The lowest point of bifurcation corresponding to the limit $k \to \infty$ (see the discussion below) is indicated on each curve in Fig. 1. The value of P/P_0 at the lowest point of bifurcation is 0·86 for $n = 3$ and 0·90 for $n = 10$, while the expansion of the hole can be computed from

$$R_a/a = \exp\left(\tfrac{1}{2} m \epsilon_a\right). \qquad (4{\cdot}1)$$

The variation of $\Phi(R)$ in the vicinity of the cavity is shown in Fig. 3 for the case $n = 3$ for $k = 10$ and $k = 100$. From (3·1) it is seen that v_R is proportional to Φ. For $k = 100$, Φ has decayed to nearly zero at distances from the cavity which are only 5 percent of its current radius; the decay length decreases with increasing k. For large k the bifurcation mode is confined to a layer in the vicinity of the surface of the cavity with k nodal lines associated with the θ variation.

In the examples studied the lowest eigenvalue for a given k decreases monotonically with increasing k. At $k = 1000$ the strain at the cavity at bifurcation for the examples in Fig. 2 is less than 0·008 above the corresponding strain for surface bifurcation of a half-space of identical material subject to uniform equal biaxial stressing. Locally at the surface of the cavity the stress state is one of equal biaxial stressing. In addition, when the wave length and radial decay length of the mode are exceedingly short, the surface mode is not (locally) affected by either the curvature of the cavity or the radial gradient of the primary solution. The result for the half-space is derived in the next section. For the hyperelastic material (2·16)–(2·18) the effective strain at which the half-space first admits surface bifurcations was found to satisfy (see Section 5)

$$m\epsilon[1-\exp(-\tfrac{3}{2}m\epsilon)] = \frac{1}{3}+\frac{E_t}{E_s}, \qquad (4\cdot2)$$

where, again, $m = 1$ for biaxial tension and $m = -1$ for biaxial compression. The effective strain at bifurcation from (4·2) is plotted against $1/n$ in Fig. 4 for the uniaxial curve (2·6) with $\alpha = \frac{3}{7}$ and $\epsilon_0 = 0{\cdot}003$. For small ϵ_0 the bifurcation strain from (4·2) is essentially independent of ϵ_0 and α. Neglecting the linear elastic contribution to the uniaxial curve, that is taking $\epsilon = K\sigma_e^n$, we replace (4·2) by

$$m\epsilon[1-\exp(-\tfrac{3}{2}m\epsilon)] = \frac{1}{3}+\frac{1}{n}, \qquad (4\cdot3)$$

and the prediction from (4·3) is indistinguishable from (4·2) in the plots of Fig. 4.

Numerical results for the compressive case ($m = -1$) also display the asymptotic approach to the half-space result (4·2) as $k \to \infty$ and this limit is shown in Fig. 4. (Here again, for small ϵ_0, (4·3) with $m = -1$ gives results which are virtually identical to (4·2).) The separation between the critical value of ϵ_a at low values of k and the limit for large k from (4·2) is similar to that shown for the tensile case in Fig. 2. For the example of Fig. 2 with $k = 2$, $\epsilon_a = 1{\cdot}92$ for $n = 3$ and $\epsilon_a = 2{\cdot}03$ for $n = 10$ in compression.

From our study of the deformation theory (hyperelastic) material characterized by a monotonically increasing true stress–strain curve typical of common polycrystalline metals (2·6), we conclude that bifurcation first occurs when the strain at the surface of the cavity reaches the level needed to give rise to surface bifurcation in the analogous half-space problem. That conclusion can be supported by a simple asymptotic surface expansion for large k. Essentially infinitely many short-wavelength modes are available just above this lowest bifurcation point. We expect that such bifurcations occur for a more general class of materials than we have studied in this paper, but we have not obtained general conditions governing such occurrences.

We conclude the discussion of the deformation theory cavity problem by emphasizing that the surface-like bifurcations discussed above take place within the elliptic regime of the governing partial differential equations. Since hydrostatic pressure or tension does not affect the transition from ellipticity to hyperbolicity in the present incompressible material, the state of stress in the primary solution is everywhere equivalent to a uniaxial state of stress σ, where σ is negative for the tensile case ($m = 1$) and positive for the compression case ($m = -1$). Let the uniaxial axis coincide with the 1-axis in a Cartesian system. It can be shown (details will be given elsewhere) that an incompressible isotropic, hyperelastic solid subject to a uniaxial stress σ lies within the elliptic range if

$$\sigma^2 < (\mathscr{L}_{2323}+E_t)\,(4\mathscr{L}_{1212}-\mathscr{L}_{2323}-E_t). \tag{4·4}$$

Here, the instantaneous moduli are those in the relation $\overset{\triangledown}{\mathbf{s}} = \mathscr{L}\cdot\cdot\mathbf{D}$ and E_t is the instantaneous tangent modulus governing a uniaxial increment in the 1-direction. Shear bands first become possible and hyperbolicity sets in when the inequality reverses in (4·4).

For the deformation theory material (2·16)–(2·18), the ellipticity condition (4·4) specializes to

$$\sigma^2 < \tfrac{1}{9}[E_s+3E_t]\,[(4q-1)\,E_s-3E_t], \tag{4·5}$$

where

$$q = (3\epsilon/2)\coth(3\epsilon/2). \tag{4·6}$$

For a pure power relation $\epsilon = K\sigma_e^n$, i.e. with the linear term in (2·6) neglected, the effective strain at the transition from ellipticity is obtained from (4·5) as

$$\epsilon^2 = \frac{1}{9}\left(1+\frac{3}{n}\right)\left(4q-1-\frac{3}{n}\right). \tag{4·7}$$

The strain from (4·7) is shown in Fig. 4 as a function of $1/n$. It is essentially identical to the result from (4·5) for the full Ramberg–Osgood curve. Since the largest value of ϵ is attained at the surface of the cavity, both the lowest tensile and compressive bifurcations occur within the elliptic range, although the strain at the cavity in the tensile case is not far from transition.

5. *Surface bifurcations on a half-space subject to equal biaxial stressing.* In the current primary state let x_i ($i = 1, 3$) be a Cartesian set of axes with the half-space occupying $x_1 \leqslant 0$. These axes will be used in the subsequent analysis. The primary state whose uniqueness is in question is a homogeneous state of equal biaxial stressing parallel to the traction-free surface, $x_1 = 0$, so that $\sigma_{22} = \sigma_{33} = \sigma$ are the non-zero stress components. The material is taken to be incompressible and its incremental moduli at the current state are uniform and transversely isotropic with respect to the x_1-axis. The Jaumann rate of the Cauchy stress is related to the Eulerian strain-rate by

$$\overset{\triangledown}{\sigma}_{ij} = \mathscr{L}_{ijkl}D_{kl}+g\delta_{ij} \quad (D_{jj} = 0). \tag{5·1}$$

In addition to the indicial symmetries $\mathscr{L}_{ijkl} = \mathscr{L}_{jikl} = \mathscr{L}_{ijlk}$, it is assumed that $\mathscr{L}_{ijkl} = \mathscr{L}_{klij}$. Transverse isotropy requires

$$\mathscr{L}_{2222} = \mathscr{L}_{3333}, \tag{5·2}$$

$$\mathscr{L}_{1122} = \mathscr{L}_{1133}, \tag{5·3}$$

$$\mathscr{L}_{1212} = \mathscr{L}_{1313} \tag{5·4}$$

and

$$2\mathscr{L}_{2323} = \mathscr{L}_{2222}-\mathscr{L}_{2233}. \tag{5·5}$$

Components such as $\mathscr{L}_{1211}$ and $\mathscr{L}_{1213}$ vanish. The above incremental relation describes the hyperelastic material (2·16)–(2·18) in the state of equal biaxial stressing. It also characterizes the plastic loading branch of the flow theory (2·1)–(2·3) and the hypo-elastic material (2·19) in this state of stress.

The components of the Eulerian strain-rate are related to the velocity components by

$$D_{ij} = \tfrac{1}{2}(v_{i,j} + v_{j,i}). \tag{5·6}$$

Incremental equilibrium is most easily stated in terms of the nominal stress-rate $\dot{n}_{ij}$ as

$$\dot{n}_{ij,i} = 0. \tag{5·7}$$

Using the connection between the nominal stress-rate and the Jaumann rate of the Cauchy stress together with (5·1) and (5·6), one can show that

$$\dot{n}_{ij} = c_{ijkl} v_{l,k} + g\delta_{ij}, \tag{5·8}$$

where

$$c_{ijkl} = \mathscr{L}_{ijkl} + \tfrac{1}{2}\sigma_{ik}\delta_{lj} - \tfrac{1}{2}\sigma_{il}\delta_{kj} - \tfrac{1}{2}\sigma_{jk}\delta_{li} - \tfrac{1}{2}\sigma_{jl}\delta_{ki}. \tag{5·9}$$

The incremental equations for the half-space admit bifurcation modes of the form

$$\left.\begin{aligned} v_1 &= A_1\, e^{\rho x_1} \cos\rho_2 x_2 \cos\rho_3 x_3, \\ v_2 &= A_2\, e^{\rho x_1} \sin\rho_2 x_2 \cos\rho_3 x_3, \\ v_3 &= A_3\, e^{\rho x_1} \cos\rho_2 x_2 \sin\rho_3 x_3, \\ g &= G\, e^{\rho x_1} \cos\rho_2 x_2 \cos\rho_3 x_3, \end{aligned}\right\} \tag{5·10}$$

where ρ_2 and ρ_3 are real and ρ is complex with positive real part such that the mode decays exponentially into the half-space ($x_1 \leqslant 0$). The amplitude factors, the A_i and G, may also be complex. It is to be understood here and below that physical quantities are given by the real part of a complex value. On the surface $x_1 = 0$ the normal velocity component varies in a checkerboard pattern according to $\cos\rho_2 x_2 \cos\rho_3 x_3$; the other components are phased in a manner consistent with the normal component. The incompressibility condition $v_{i,i} = 0$ requires

$$\rho A_1 + \rho_2 A_2 + \rho_3 A_3 = 0. \tag{5·11}$$

Using (5·8)–(5·11) together with (5·2)–(5·4), one can reduce the three equations of incremental equilibrium (5·7) to the algebraic equations

$$\rho A_1[(\mathscr{L}_{1111} - \mathscr{L}_{1122} - \mathscr{L}_{1212} + \sigma/2)\,\rho^2 - (\mathscr{L}_{1212} + \sigma/2)\,(\rho_2^2 + \rho_3^2)] = -G\rho^2, \tag{5·12}$$

$$\begin{aligned} &\rho_2 A_2[(\mathscr{L}_{1212} - \sigma/2)\,\rho^2 + (\mathscr{L}_{1212} - \mathscr{L}_{2222} + \mathscr{L}_{1122} + \sigma/2)\,\rho_2^2 - \mathscr{L}_{2323}\rho_3^2)] \\ &\qquad + \rho_3 A_3[\mathscr{L}_{1212} - \mathscr{L}_{2233} + \mathscr{L}_{1122} - \mathscr{L}_{2323} + \sigma/2]\,\rho_2^2 = G\rho_2^2, \end{aligned} \tag{5·13}$$

$$\begin{aligned} &\rho_3 A_3[(\mathscr{L}_{1212} - \sigma/2)\,\rho^2 + (\mathscr{L}_{1212} - \mathscr{L}_{2222} + \mathscr{L}_{1122} + \sigma/2)\,\rho_3^2 - \mathscr{L}_{2323}\rho_2^2] \\ &\qquad + \rho_2 A_2[\mathscr{L}_{1212} - \mathscr{L}_{2233} + \mathscr{L}_{1122} - \mathscr{L}_{2323} + \sigma/2]\,\rho_3^2 = G\rho_3^2 \end{aligned} \tag{5·14}$$

supplemented by (5·11).

To reduce these equations further, add (5·13) and (5·14), noting (5·5) and (5·11) with the result

$$\rho A_1[(\mathscr{L}_{1212}-\sigma/2)\rho^2+(\mathscr{L}_{1212}+\mathscr{L}_{1122}-\mathscr{L}_{2222}+\sigma/2)(\rho_2^2+\rho_3^2)] = -G(\rho_2^2+\rho_3^2). \quad (5\cdot15)$$

Next, let

$$z = \rho/p \quad \text{where} \quad p = \sqrt{(\rho_2^2+\rho_3^2)} \quad (5\cdot16)$$

and then rewrite the two equations (5·12) and (5·15) involving the two unknown amplitude factors A_1 and G as

$$\rho A_1[(\mathscr{L}_{1111}-\mathscr{L}_{1122}-\mathscr{L}_{1212}+\sigma/2)z^2-(\mathscr{L}_{1212}+\sigma/2)]+Gz^2 = 0, \quad (5\cdot17)$$

$$\rho A_1[(\mathscr{L}_{1212}-\sigma/2)z^2+(\mathscr{L}_{1212}+\mathscr{L}_{1122}-\mathscr{L}_{2222}+\sigma/2)]+G = 0. \quad (5\cdot18)$$

Existence of a solution to (5·17) and (5·18) requires

$$(\mathscr{L}_{1212}-\sigma/2)z^4-(\mathscr{L}_{1111}+\mathscr{L}_{2222}-2\mathscr{L}_{1122}-2\mathscr{L}_{1212})z^2+(\mathscr{L}_{1212}+\sigma/2) = 0 \quad (5\cdot19)$$

Since there is no characteristic length in the half-space problem, only the *ratio* of the wave numbers, in the form (5·16), can be determined. Arbitrary real values may be assigned to ρ_2 and ρ_3; ρ is then obtained from z using (5·16), as will be made clearer in the sequel. By considering the difference between equations (5·13) and (5·14) one can show that A_2 and A_3 are given in terms of A_1 by

$$A_2 = -A_1\rho\rho_2/p^2 \quad \text{and} \quad A_3 = -A_1\rho\rho_3/p^2. \quad (5\cdot20)$$

Thus A_1 may be regarded as the amplitude of the eigenmode with A_2, A_3 and G tied to it by (5·20) and (5·17) or (5·18).

In what follows, it will be useful and illuminating to introduce two instantaneous shearing moduli μ and μ^* which will now be defined. In the current state of equal biaxial stressing, consider a 'plane strain' increment with $D_{33} = 0$ (or equivalently with $D_{22} = 0$) and note that (5·1) gives

$$\overset{\triangledown}{\sigma}_{12} = 2\mu D_{12} \quad \text{and} \quad \overset{\triangledown}{\sigma}_{11}-\overset{\triangledown}{\sigma}_{22} = 2\mu^*(D_{11}-D_{22}), \quad (5\cdot21)$$

where

$$\mu = \mathscr{L}_{1212} \quad \text{and} \quad 4\mu^* = \mathscr{L}_{1111}+\mathscr{L}_{2222}-2\mathscr{L}_{1122}. \quad (5\cdot22)$$

Thus μ governs an increment of shear parallel to the x_1 axis and any axis perpendicular to it, while μ^* is the instantaneous shear modulus for shearing at 45° to those axes. It is also seen that $4\mu^*$ is the instantaneous plane strain tangent modulus such that $\overset{\triangledown}{\sigma}_{11} = 4\mu^* D_{11}$ with $D_{33} = 0$ and $\overset{\triangledown}{\sigma}_{22} = 0$. These moduli are featured in the plane strain incremental analysis of Hill and Hutchinson (3) and of Young (4).

The coefficients of the quadratic in z^2 in (5·19) can be expressed in terms of μ and μ^*. The lowest bifurcation occurs at a stress satisfying $\sigma^2 < 16\mu^*(\mu-\mu^*)$, as can be verified a posteriori, such that there are two roots of (5·19) with positive real parts which are complex conjugates of one another. Denote the square root with positive real part of

$$z^2 = \frac{2\mu^*-\mu+i\sqrt{\{4\mu^*(\mu-\mu^*)-\sigma^2/4\}}}{(\mu-\sigma/2)} \quad (5\cdot23)$$

by z and its conjugate by $\bar{z}$ where $i = \sqrt{-1}$. The general solution of the form (5·10) to the incremental equilibrium equation satisfying exponential decay as $x_1 \to -\infty$ is

$$\left.\begin{aligned} v_1 &= (A\,e^{\bar{\rho}x_1} + \bar{A}\,e^{\rho x_1})\cos\rho_2 x_2 \cos\rho_3 x_2, \\ v_2 &= -\rho_2 p^{-2}(A\rho\, e^{\rho x_1} + \bar{A}\bar{\rho} e^{\bar{\rho}x_1})\sin\rho_2 x_2 \cos\rho_3 x_3, \\ v_3 &= -\rho_3 p^{-2}(A\rho\, e^{\rho x_1} + \bar{A}\bar{\rho}\, e^{\bar{\rho}x_1})\cos\rho_2 x_2 \sin\rho_3 x_3, \\ g &= (G\,e^{\rho x_1} + \bar{G}\,e^{\bar{\rho}x_1})\cos\rho_2 x_2 \cos\rho_3 x_3, \end{aligned}\right\} \tag{5·24}$$

where A ($\equiv A_1$) is complex, $\rho = zp$ and G is given in terms of A by (5·17) or (5·18).

A zero traction-rate on the free surface $x_1 = 0$ requires $n_{1i} = 0$ ($i = 1, 3$); these three equations are satisfied by the eigenmode (5·24) if

$$(\mathscr{L}_{1111} - \mathscr{L}_{1122})(A\rho + \bar{A}\bar{\rho}) + G + \bar{G} = 0, \tag{5·25}$$

$$A(1+z^2) + \bar{A}(1+\bar{z}^2) = 0. \tag{5·26}$$

Use (5·18) to eliminate $G + \bar{G}$ in (5·25) in favour of A and $\bar{A}$; the resulting equation and (5·26) have a solution if and only if

$$(\mu - \tfrac{1}{2}\sigma)(z^2\bar{z}^2 + z^2 + \bar{z}^2 + z\bar{z}) + (\mu - 4\mu^* + \sigma/2)(1 - z\bar{z}) = 0. \tag{5·27}$$

This equation can be reduced further using (5·23) to the bifurcation condition

$$\frac{\sigma}{4\mu^*} = 1 + \frac{\sigma}{4\mu^*}\bigg/\sqrt{\left(\frac{2\mu - \sigma}{2\mu + \sigma}\right)}. \tag{5·28}$$

An equation equivalent to (5·28) was derived for the special case of incremental plane strain by Biot (10), (4·39). This equation, with precisely the same notation, was also obtained for the plane problem by Hill and Hutchinson (3), (6·5), and by Young (4), (5·28), who specifically considered the half-space problem. The family of modes (5·24) includes plane strain modes (e.g. with $\rho_3 = 0$, $v_3 = 0$ and the velocity vector lies within the x_1–x_2 plane). But it also includes inherently three dimensional modes with arbitrary values of ρ_2 and ρ_3, all of which are associated with the same bifurcation stress (5·28). As already remarked, the lateral scale of the modes is undetermined since there is no characteristic length in the problem; the exponential decay into the interior is related to the lateral scales through (5·16). It is also noted that ρ_2 and ρ_3 affect the exponential decay only in combination $\rho_2^2 + \rho_3^2$, and thus all possible aspect ratios of the lateral checkerboard variation are simultaneously available for each possible ρ. This multiplicity stems from the incremental transverse isotropy in the primary state of equal biaxial stressing.

Axisymmetric surface modes also first become possible when (5·28) is attained. With $R = (x_2^2 + x_3^2)^{\frac{1}{2}}$, the two non-zero velocity components in a cylindrical coordinate system are related to a stream function $\chi(x_1, R)$ by

$$v_R = -\partial\chi/\partial x_1, \quad v_1 = R^{-1}\,\partial(R\chi)/\partial R \tag{5·29}$$

as a consequence of incompressibility. By carrying out an analysis along the lines of the one just completed, it can be shown that (5·28) is the eigenvalue equation for modes of the form

$$\chi = (c\,e^{\lambda z x_1} + \bar{c}\,e^{\lambda \bar{z} x_1})\,J_1(\lambda R), \tag{5·30}$$

where c is a complex amplitude factor, λ is an arbitrary positive scale factor, z is defined as before in (5·23), and J_1 is the Bessel function of the first order of the first kind.

For the deformation theory solid, (2·16)–(2·18),

$$\mu = \tfrac{1}{3}qE_s \quad \text{and} \quad 4\mu^* = E_t + E_s/3, \tag{5·31}$$

where $q = (3\epsilon/2)\coth(3\epsilon/2)$ and ϵ is the effective strain defined in (2·14). When expressed in terms of ϵ, the bifurcation condition (5·28) becomes (4·2). Specializing further to a pure power relation between effective stress and effective strain, i.e. $\epsilon = K\sigma_e^n$, one finds $E_s/E_t = n$ with (4·2) replaced by (4·3). For large n (4·3) gives

$$\epsilon = 0{\cdot}576 + 1{\cdot}06/n + O(1/n^2) \quad (m = 1, \text{ biaxial tension}),$$

$$\epsilon = 0{\cdot}402 + 0{\cdot}518/n + O(1/n^2) \quad (m = -1, \text{ biaxial compression}),$$

with the complete results plotted in Fig. 4.

For the flow theory solid, (2·1)–(2·3),

$$\mu = E/3 \quad \text{and} \quad 4\mu^* = E_t + E/3. \tag{5·32}$$

From (5·28) it is readily seen that bifurcation in a surface mode of the type considered here requires σ to be on the order of the Young's modulus E (roughly $E/2$ if $E_t \ll E$). A stress level of this magnitude is normally far beyond attainable levels in polycrystalline metals.

The sensitivity of the onset of surface bifurcations to the type of elastoplastic constitutive relation used is further demonstrated by considering the hypoelastic material (2·19) for which

$$\mu = \tfrac{1}{3}E_s \quad \text{and} \quad 4\mu^* = E_t + E_s/3. \tag{5·33}$$

Now the bifurcation condition (5·28) reduces to

$$m\epsilon\left[1 - \sqrt{\frac{1 - 3m\epsilon/2}{1 + 3m\epsilon/2}}\right] = \frac{1}{3} + \frac{E_t}{E_s} \tag{5·34}$$

instead of (4·2). For the pure power law relation, $E_s/E_t = n$ and for large n (5·34) yields

$$\epsilon = 0{\cdot}517 + 0{\cdot}748/n + O(1/n^2) \quad (m = 1, \text{ biaxial tension}),$$

$$\epsilon = 0{\cdot}375 + 0{\cdot}409/n + O(1/n^2) \quad (m = -1, \text{ biaxial compression}).$$

These eigenstrains are about 10 % below the corresponding predictions for the hyper-elastic deformation theory. It was noted (cf. Fig. 4) that the lowest surface bifurcations occur within the elliptic range for $1 \leqslant n < \infty$. This is not the case for the hypoelastic material. In fact, for the tensile case ($m = 1$), for example, (5·34) admits a solution only for $n \geqslant 3$. At $n = 3$, $\epsilon = \tfrac{2}{3}$ which coincides with loss of ellipticity (Needleman and Rice (11)).

6. *Discussion.* In an elastoplastic medium, bifurcation usually takes place under increasing overall straining, involving a combination of the eigenmode and the primary solution, such that the instantaneous moduli associated with plastic loading

are everywhere in effect. The relevance of the deformation theory (or of the hypoelastic relation (2·19)) to elastoplastic solids is that it models the instantaneous moduli associated with nearly proportional loading when a corner is assumed to develop on the yield surface. The combination of primary solution increment and eigenmode comprising the total bifurcation mode is constrained to ensure that nearly proportional loading does occur everywhere at bifurcation (Hill (2)).

A yield surface corner as modelled by the deformation theory substantially reduces certain of the incremental moduli as compared to the flow theory based on a smooth yield surface (2·3). This accounts for the fact that bifurcation occurs for the deformation theory solid and for the solid (2·19) but that it is effectively excluded as a possibility for the flow theory solid (2·3). The issue of which class of theories best models polycrystalline elastoplastic solids in bifurcation applications is a complicated one which is not yet fully resolved. In applications involving plastic buckling of plates and necking in thin sheets under biaxial tensile straining, bifurcation predictions based on deformation theory do appear to be more relevant than corresponding predictions from classical flow theory. The bifurcations at a spherical hole found here take place at much larger strains than is usual in plastic buckling and at somewhat larger strains than in sheet necking problems. The physical significance of the bifurcations found here remains to be seen.

Another unusual feature of the present problem is that the characteristic wave length of the lowest bifurcation mode remains undetermined and must only be very short compared to the radius of the hole. There is nothing in the structure of the continuum elastoplastic theory used here which limits the spatial gradients of the strain-rates. It is tacitly assumed that the scale of the deformation is long compared to grain size, for example, but no grain size effect is incorporated in any of the present constitutive laws. It is possible, under some circumstances, that the scale of the surface bifurcations found here is determined by grain size considerations and that the bifurcations play a role in the development of roughness on the surfaces of highly strained polycrystalline metals.

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