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THE FORCE EXERTED BY A MOVING ICE SHEET ON AN OFFSHORE STRUCTURE

PART I. THE CREEP MODE

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ABSTRACT

A uniform ice sheet moves steadily against a flat-sided fixed offshore structure. The ice deforms in creep: the problem is to determine the relation between the force on the structure and the velocity of the ice. The problem cannot be solved analytically, but a convenient approximate solution can be found by the reference stress method of creep structural analysis, which makes systematic use of a small number of numerical and analytic solutions to related problems, including solutions to the corresponding problem for a perfectly-plastic material. The method is applicable to any observed relation between stress and strain rate. It is tested by comparison with additional numerical solutions, and with published data on indentation experiments.

INTRODUCTION

Extremely complex deformations occur when floating ice interacts with offshore structures. The ice deforms elastically, creeps, yields and breaks, and the broken fragments pile up in front of the structure and are carried round it. It is important to know how to calculate the forces that ice exerts on a structure in this situation. A full understanding of the problem is still some way off. This paper sets out an analysis of one aspect of the problem, and analyses the forces that are set up when a slow-moving floating ice sheet moves against a structure. The ice moves slowly enough for all the deformation to be by creep; fracture does not occur. The geometry is a simple one (Fig. 1): a straight-sided uniform ice sheet comes into contact with the straight vertical face of a rigid structure, advancing at right angles to it with a uniform velocity.

This is obviously a severe idealisation of the complex geometry and non-uniform material properties that occur in the field. Creep will generally be the dominant mode of deformation only when the ice is moving very slowly. An analysis of the creep problem is worthwhile because it represents one limiting case. When the ice velocity is high, the actual ice force will generally be smaller than the force corresponding to this limiting case, because of the effects of fracture (Palmer et al., 1983).

MATERIAL IDEALISATION

An analysis requires an idealisation of the material properties of ice. At strain rates up to 10^{-4} s^{-1} in compression, and up to 10^{-5} s^{-1} in tension, the deformation of ice is dominated by creep. At higher strain rates, fracture becomes dominant. Within the creep range, creep strains are generally large by com-

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parison with the elastic strains that also occur: if ice at $-7°C$ is loaded in uniaxial compression at $2\ MN/m^2$ ($290\ lb/in^2$), a typical stress close to the contact with an offshore structure, the creep strain after 10 min is 10^{-3} (Gold, 1977), five times the elastic strain. The creep strain rate $\dot{\epsilon}$ depends strongly on stress, and is observed to be proportional to the nth power of the stress, where n is about 3 (Glen's law). At strain rates above $10^{-5}\ s^{-1}$, the power law breaks down and leads to an even more rapid increase of strain rate with increasing stresses.

In the present study, ice is idealised as an isotropic creeping material whose instantaneous strain rate is a function of the stress, but not of the strain or the previous loading history. Under uniaxial loading, the stress σ corresponding to a strain rate $\dot{\epsilon}$, and the stress σ_0 corresponding to a strain rate $\dot{\epsilon}_0$, are in general related by

$$\dot{\epsilon}/\dot{\epsilon}_0 = f(\sigma/\sigma_0) \,. \tag{1}$$

In the power-law creep regime, for instance

$$\dot{\epsilon}/\dot{\epsilon}_0 = (\sigma/\sigma_0)^n \,. \tag{2}$$

It is helpful to consider two extreme cases of power-law creep. If n tends to infinity, eqn. (2) becomes the stress–strain-rate relation for a rigid-plastic material with yield stress σ_0 (so that the strain rate can only be non-zero when σ reaches σ_0), whereas n equal to 1 corresponds to a linear-viscous material (and also would correspond to a linear-elastic material if the strain rate $\dot{\epsilon}$ were replaced by the strain ϵ, and $\sigma_0/\dot{\epsilon}_0$ were identified as the Young's modulus).

The stress–strain-rate relation can be generalised to describe multiaxial stress. The most straightforward generalisation of the power-law stress–strain-rate relation (Palmer, 1967) is

$$\dot{\epsilon}_{ij} = \frac{3}{2}\,\frac{\dot{\epsilon}_0}{\sigma_0^n}\left\{\frac{3}{2}\,s_{kl}\,s_{kl}\right\}^{(n-1)/2} s_{ij} \tag{3}$$

where $\dot{\epsilon}_{ij}$ is the strain rate tensor, s_{ij} the deviatoric stress tensor, and the repeated subscript summation notation is used. This generalisation is widely used in glaciology and ice mechanics. The extreme values of n again correspond to a rigid-plastic material obeying a von Mises yield condition ($n \to \infty$), and to an incompressible linear viscous material ($n = 1$).

It has to be recognised that the stress–strain-rate relation (3) is an idealization. Sea ice is often anisotropic, while its creep rate may depend on the first and third invariants of the deviatoric stress tensor as well as the second invariant $s_{kl}s_{kl}$. The analysis described below can be generalised to take account of these effects.

METHODS OF SOLUTION

The problem is to find the force between an ice sheet and a structure, when the ice moves steadily towards the structure and deforms by steady-state creep described by the material stress–strain-rate relation (3). The extent of deformation is sufficiently small for the geometry not to alter significantly.

Applied mechanics offers at least five approaches to this problem. They are:

(A) through a complete solution, combining the differential equations of compatibility and equilibrium, and the non-linear stress–strain-rate relation. Some indentation problems have known closed-form solutions if the material is linear, but the non-linear problem is out of reach of analytical solution;

(B) through a non-linear finite-element or finite-difference formulation, leading to a numerical solution;

(C) through bound methods (Martin, 1966; Palmer, 1967; Ponter, 1970), which make it possible to use relatively straightforward calculations to set upper and lower bounds on the total force corresponding to a certain relative velocity. These methods use postulated stress and strain rate distributions, and are related to variational methods in elasticity;

(D) through the reference-stress concept (Anderson et al., 1963; Marriott, 1970), which makes it possible to arrive at approximate solutions to non-linear structural problems;

(E) through an approximate solution based on similarity and dimensional analysis.

This study concentrates on the reference-stress method, which is attractively simple and gives analytical expressions which can be applied to a wide range of geometries and material properties.

REFERENCE STRESS THEORY

The origin of the reference-stress method was in the engineering analysis of structures subject to creep.

Its objective is to correlate creep deformations in a structure directly with an equivalent creep test. If a structure is statically determinate, the loads determine the stresses throughout the structure (independently of the stress–strain relation). All that is then required to calculate the deformation rate of the structure is to determine stresses under operating loads, to carry out creep tests at those stresses to determine the corresponding strain rates, and finally to apply standard methods to calculate the deformation rates from the creep strain rates. Indeterminate structures are much more difficult to analyse, because the stresses themselves depend on the stress–strain-rate relation, which may be complex and uncertain. That suggests the following question: at what stress ought one to measure the strain rate of the material, if one wishes the calculated deformation rate for the structure to be as little sensitive as possible to the details of the stress–strain-rate relation? Among the primary motivations for development of the reference-stress method were experimental uncertainty about the power-law exponent n and the need for calculations for structures that operate beyond the threshold of power-law breakdown.

The idea can be taken further: if an exact or an approximate solution to a creep problem can be written in a form which is insensitive to the details of the stress–strain-rate relation (to the power-law exponent n, for instance), it becomes possible to incorporate into the solution at least some of the results from solutions to the same problem for related but simpler stress–strain-rate relations, such as those from linear elasticity and perfect plasticity. The reference-stress method does this in a systematic way. The results are approximate, and are not usually either upper or lower bounds, but the method is very simple, and comparison with exact solutions shows that the results are accurate enough for engineering applications. The method also provides a systematic method for making the results of numerical calculations applicable to a wider family of problems.

Imagine a general structure subjected to a single load P, and let the velocity of the point of application of the load be U; the rest of the structure boundary is either free of external loads or fixed to an immovable foundation. The rate at which the load does work on the structure is PU, and can be equated to the rate at which work is dissipated internally

$$PU = \int \sigma_{ij}\dot{\epsilon}_{ij}\,\mathrm{d}V \tag{4}$$

integrated over the whole structure, where $\sigma_{ij}\dot{\epsilon}_{ij}$ is the rate of dissipation per unit volume, the rate at which the stress σ_{ij} does work on the strain rate $\dot{\epsilon}_{ij}$, and of course varies from point to point. The right-hand side is now reqritten so that

$$PU = \sigma_R\, F_{\dot{\epsilon}}[\sigma_R]\, V_R\,, \tag{5}$$

where σ_R is a reference stress, $F_{\dot{\epsilon}}[\sigma_R]$ is the strain rate that corresponds to the reference stress* (through the material stress–strain-rate relation), and V_R is a reference volume (and can loosely be thought of as an equivalent volume over which the deformation occurs). The reference stress is chosen so that

$$\sigma_R/P = Y/P_L\,, \tag{6}$$

where P_L is the limit load that corresponds to plastic collapse of a structure with the same geometry but made of a perfectly-plastic material with yield stress Y. In other words, the ratio of the reference stress to an (arbitary) yield stress is the same as that of the applied force to the limit load corresponding to that yield stress. The use of the limit load corresponding to a yield stress is no more than a device to find the reference stress, and it is not implied that the material is perfectly-plastic.

The method can be illustrated and tested by applying it to a problem which has a known analytic solution. Consider a thick-walled cylindrical tube (internal radius a, external radius λa), composed of a material which creeps, and loaded by internal pressure, under plane-strain conditions, so that axial deformation is prevented. We wish to estimate the velocity U at which the internal radius increases. The work equation corresponding to (4) is

$$2\pi a p U = \sigma_R F_{\dot{\epsilon}}[\sigma_R]\, V_R \tag{7}$$

per unit length of tube, where p is the internal pressure. The limit pressure for a thick-walled tube is $(2/\sqrt{3})Y \ln \lambda$ (Calladine, 1969), for a von Mises material with a yield stress Y in uniaxial tension. From eqn. (6), the ratio of the reference stress to the pressure is the same as that of the limit pressure to the yield stress Y, and so

$$\sigma_R = p/((2/\sqrt{3})\ln \lambda)\,. \tag{8}$$

*Throughout this paper, $F_{\dot{\epsilon}}[x]$ means the uniaxial strain rate that corresponds to a uniaxial stress x, and $F_\sigma[x]$ means the uniaxial stress that corresponds to a uniaxial strain rate x; square brackets are not used in any other context.

An elementary elastic solution shows that for a tube made from a linear-elastic incompressible material, the radial displacement U induced by a pressure p is

$$U = \tfrac{3}{2} pa/(1-\lambda^{-2})E . \tag{9}$$

Here, U and E are to be interpreted as the radial displacement and the Young's modulus for an elastic material, and as the velocity and the ratio between uniaxial stress and strain rate for a linear viscous material. The strain rate $F_{\dot\epsilon}[\sigma_R]$ corresponding to the reference stress σ_R is then σ_R/E. Substituting into (7)

$$2\pi ap \frac{3/2}{1-\lambda^{-2}} \frac{pa}{E} = \frac{p}{\frac{2}{3}\ln\lambda} \frac{p}{\frac{2}{3}E\ln\lambda} V_R \tag{10}$$

and so

$$V_R = 2\pi(\ln\lambda)^2 a^2/(1-\lambda^{-2}) . \tag{11}$$

These values of V_R and σ_R can now be used *generally*, for any stress–strain-rate relation, and (7) becomes

$$\frac{U}{a} = \frac{(\sqrt{3}/2)\ln\lambda}{1-\lambda^{-2}} F_{\dot\epsilon}\left[\frac{p}{(2/\sqrt{3})\ln\lambda}\right] \tag{12}$$

a numerical function of λ multiplying the strain rate that corresponds to the reference stress. If the relation between strain rate and stress is known from an experiment, eqn. (12) can be used directly to determine the velocity U, and it tells us that the appropriate uniaxial stress for a creep test is the reference stress $p/(2\sqrt{3})\ln\lambda$.

If stress and strain rate are related by the power-law relation (2), eqn. (12) becomes

$$\frac{U}{a} = \frac{3/2}{1-\lambda^{-2}}\left(\frac{2}{\sqrt{3}}\ln\lambda\right)^{1-n} \dot\epsilon_0\,(p/\epsilon_0)^n \tag{13}$$

and so the pressure is related to the velocity U by

$$p = \left(\frac{1-\lambda^{-2}}{3/2}\right)^{1/n}\left(\frac{2}{\sqrt{3}}\ln\lambda\right)^{1-1/n} \sigma_0(U/a\dot\epsilon_0)^{1/n} \tag{14}$$

The exact solution to the problem is

$$p = \left(\frac{2}{\sqrt{3}}\right)^{1/n} \frac{n}{\sqrt{3}}(1-\lambda^{-2/n})\,\sigma_0(U/a\dot\epsilon_0)^{1/n} . \tag{15}$$

Table 1 compares the reference-stress solution (14) to the exact solution (15). It can be seen that the agreement is encouragingly good, and that the maximum discrepancy is never more than 2 per cent.

TABLE 1

Comparison between reference stress and exact solutions (Thick-walled cylinder (external radius/internal radius) = 2.)

Power-law exponent, n	Dimensionless internal pressure to velocity ratio, $p/\sigma_0(U/a\dot\epsilon_0)^{1/n}$ reference stress	exact
1	0.5000	0.5000
1.5	0.5849	0.5749
2	0.6326	0.6204
3	0.6842	0.6724
4	0.7116	0.7012
5	0.7285	0.7194
10	0.7636	0.7582
∞	0.8004	0.8004

APPLICATION TO ICE FORCE PROBLEM

The method can now be applied to the ice force problem, in which a semi-infinite ice sheet of thickness t moves at a velocity U against a structure and is in contact over a width D (Fig. 1). The relative velocity between the structure and the ice is then U, and eqn. (5) expresses the rate at which the ice force P between the structure and the ice does work on the ice. Substituting $P/(P_L/Y)$ for σ_R, eqn. (5) can be rewritten

$$F_{\dot\epsilon}[\sigma_R] = \frac{P_L/Y}{V_R}\frac{U}{D} \tag{16}$$

so that the strain rate corresponding to the reference stress is $(P_L/Y)U/V_RD$. It follows that the reference stress corresponds to that strain rate, and so

$$\sigma_R = F_\sigma\left[\frac{P_L/Y}{V_R}\frac{U}{D}\right] . \tag{17}$$

It is helpful to introduce dimensionless variables ϕ and δ, defined so that

$$\phi = P_L/YDt, \tag{18}$$

$$\psi = \frac{V_R/D^2t}{(P_L/YDt)^2} . \tag{19}$$

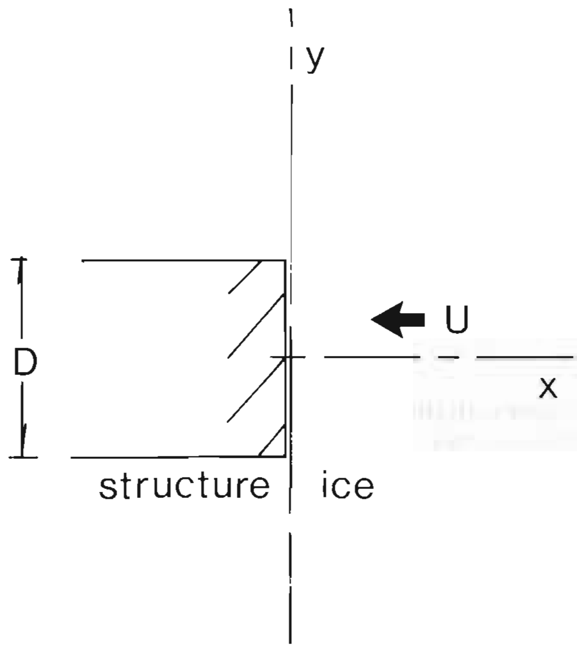

Fig. 1. Problem definition.

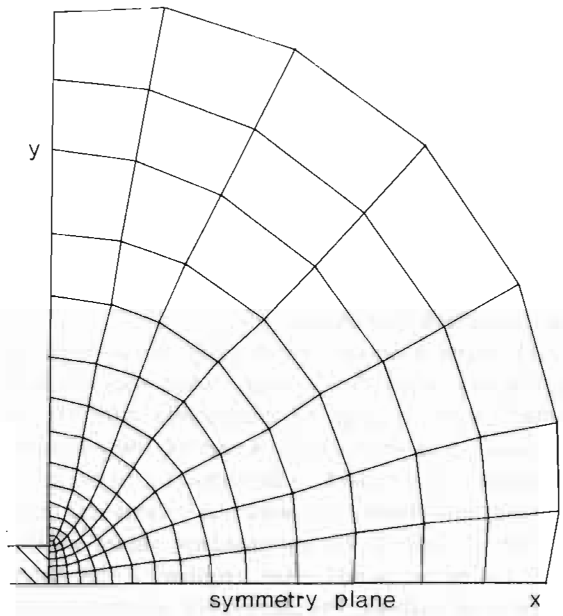

Fig. 2. Mesh for finite-element calculations.

After substituting for V_R and P_L in eqn. (17), and rearranging, it becomes

$$P = \phi D t\, F_\sigma \left[\frac{U/D}{\phi\psi}\right]. \tag{20}$$

This is a general formula for the ice force P induced by an ice velocity U.

The next step is to find the quantities ϕ and ψ, which are functions of the aspect ratio D/t of the contact between the structure and the ice. The simplest case is plane deformation, which will occur when the contact width D is small by comparison with the ice thickness t. The Prandtl solution to the plastic problem is known to be exact, and gives

$$\phi = P_L/YDt = (\pi+2)/\sqrt{3} \tag{21}$$

The solution for a von Mises material is the appropriate one, since it corresponds to the limiting case of the stress strain-rate relation (3).

In the thick-walled cylinder example, the reference volume was found from the elastic solution. Unfortunately, this cannot be done for the indentation problem, because the displacement field of the elastic solution (Sadowsky, 1928; Sneddon, 1951) includes logarithmic terms which grow indefinitely as distance from the indentor increases. A finite indentation force therefore induces an infinite relative displacement between the indentor and points in the far field. This is an awkward feature of the elastic solution only, and does not arise if n is greater than 1: it is discussed further in the Appendix.

Since the elastic solution is not available, a numerical solution for another value of n is needed to find ψ. A two-dimensional finite-element solution was carried out for n equal to 3. It made use of the initial strain method at a sequence of time intervals (Ponter and Brown, 1978). At time t, the current stresses are used to evaluate the creep strain $\dot{\epsilon}_0^c\,\delta t$ generated in a time interval δt which then yields, by elastic analysis, the change in stress $\delta\sigma_g$. This forward interpretation procedure is further refined by use of the Runge-Kutta method. The finite-element mesh and boundary conditions are shown in Fig. 2, where each quadrilateral is subdivided into two triangles with linear variation in displacement within each element. At time $t = 0$ the boundary velocity is given a constant value and the elastic/creep solution is continued until the stress rates have reduced to an acceptably low value. The steady-state solution gives a compatible strain-rate field consistent with the indentor velocity. An upper bound on the indentor load can then be evaluated by equating the rate of work done by the indentor to the total rate of creep energy dissipation throughout the finite element mesh, and that value of P was used to determine ψ, which is 0.445. It gives a value which is greater than that obtained by integration of the stresses (which are constant within each element) across the surface of the indentor; that alternative is not a bound, and was considered less reliable.

The other extreme case is plane stress, which is approached when D/t is large, when the structure contact width is large by comparison with the ice thickness: ϕ is then $2/\sqrt{3}$. A second finite-element calculation for n equal to 3 determines ψ to be 0.385.

It was not possible to carry out a numerical creep solution to find ψ for intermediate values of D/t, since to do so would have required a full three-dimensional analysis. The plane-strain and plane-stress values of ψ are close together, and since the ice force is proportional to $\psi^{-1/n}$, and n is at least 3, uncertainty about ψ for intermediate D/t has little influence on ice forces calculated from eqn. (20). However, ϕ is more sensitive to D/t: it can be determined by standard techniques in plasticity theory. The precise boundary condition at the contact between the structure and the ice influences the results. If the ice is free to slide vertically across the face of the structure ("free slip"), upper bound calculations give

$$\phi = \min \begin{cases} (\pi+2)/\sqrt{3}\,, \\ \dfrac{2}{\sqrt{3}}\left(1+\dfrac{\sqrt{2}}{4}\,\dfrac{t}{D}\right). \end{cases} \tag{22}$$

The second expression is based on a velocity field proposed by Croasdale et al. (1977), and can be refined slightly.

Plane stress conditions imply that the ice is free to move in the vertical direction, perpendicular to the ice sheet, and to slide vertically across the face of the structure. Friction and adhesion will often prevent these sliding movements and modify the deformation field in the ice. If the ice is not free to slide ("no slip"), and the shear strength of the interface with the structure is at least as large as the shear strength of the material

$$\phi = \min \begin{cases} (\pi+2)/\sqrt{3}\,, \\ \dfrac{2}{\sqrt{3}}\left(1.5+\dfrac{\sqrt{2}}{4}\,\dfrac{t}{D}\right), \\ \dfrac{2}{\sqrt{3}}\left(1+\dfrac{\sqrt{2}}{2}\left(\dfrac{t}{D}\right)^{1/2}+\dfrac{1}{4}\left(\dfrac{t}{D}\right)\right). \end{cases} \tag{23}$$

These expressions for ϕ are again upper bounds, and could be further refined.

The results are summarized in Table 2.

COMPARISON WITH ADDITIONAL NUMERICAL SOLUTIONS

The reference stress solution presented here is based on two solutions, an analytic solution for n tending to infinity and a numerical solution for n equal to 3. One test of the method is to obtain numerical solutions for additional values of n, and to compare the forces calculated from those solutions with the forces obtained from the reference stress results summarized in Table 2. Table 3 makes this comparison for n equal to 5 and 7; it proved impossible to obtain numerical solutions for higher values of n because of convergence difficulties. The table shows that agreement is satisfactory, although not as good as for the thick-cylinder example, probably because that example is kinematically determinate. As n increases, the force calculated from the numerical finite-element solution is greater than the force calculated from reference stress: that suggests that the deformation fields allowed by the finite-element analysis are overly restrictive as the strain rate distribution begins to approach, at larger n, the discontinuous strain rate distribution associated with plastic collapse.

COMPARISON WITH MEASUREMENTS

A number of investigators have carried out indentation experiments in which flat-faced indenters were driven at a constant velocity into the edges of plates of ice, and the force was measured. The reference-stress calculation implies that the relationship between $P/\phi Dt$ and $U/\phi\psi D$ should be identical with the relationship between stress- and strain-rate

TABLE 2

Reference stress theory: results summary

The ice force is

$$P = \phi D t\, F_\sigma \left[\frac{U/D}{\phi\psi} \right]$$

where D is the contact width, t is the ice thickness, U is the ice velocity, $F_\sigma[x]$ is the uniaxial stress corresponding to a uniaxial strain rate x and ϕ and ψ are given by the following:

Condition	ϕ	ψ
plane strain ($D/t \to 0$)	$(\pi+2)/\sqrt{3} = 2.986$	0.445
intermediate D/t; free slip between ice and structure	$\min \begin{cases} 2.986 \\ (2/\sqrt{3})\,(1 + \frac{1}{4}\sqrt{2}\, t/D) \end{cases}$	between 0.445 and 0.385
intermediate D/t; no slip between ice	$\min \begin{cases} 2.986 \\ (2/\sqrt{3})\,(1.5 + \frac{1}{4}\sqrt{2}\, t/D) \\ (2/\sqrt{3})\,(1 + \frac{1}{2}\sqrt{2}\,(t/D)^{1/2} + \frac{1}{4}\,(t/D)) \end{cases}$	
plane stress ($D/t \to \infty$)	$2/\sqrt{3} = 1.154$	0.385

TABLE 3

Comparison between the force from a finite-element solution (P_{FE}) and the force found by the reference-stress method (P_R)

n	P_{FE}/P_R	
	plane stress	plane strain
3	1	1
5	1.130	1.083
7	1.154	1.103

in uniaxial compression. This gives us a straightforward method of comparison between theory and experiment.

Frederking and Gold (1975) carried out edge-indentation tests on plates of columnar-grained freshwater ice at -10°C. The ice had random orientation of the c-axes parallel to the plane of the plates, and a grain diameter of 2–5 mm. The plate thickness ranged from 25 to 98 mm and the indenter widths from 13 to 150 mm; the range of D/t was from 0.21 to 3. In most of the tests the velocity U was 8.3×10^{-5} mm/s. The results allow the reference stress $P/\phi Dt$ to be calculated for each test. In Fig. 3,

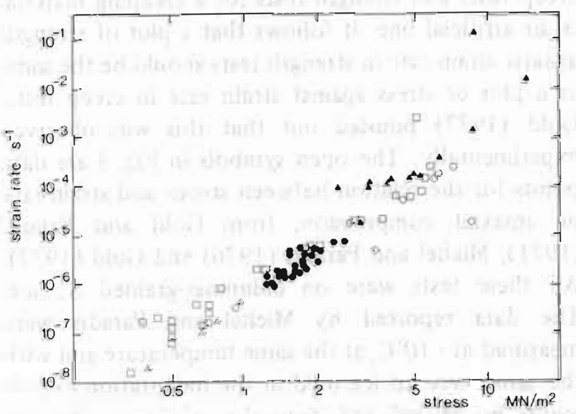

Fig. 3. Comparison between indentation tests and uniaxial tests:
Indentation: ● Frederking and Gold, ▲ Michel and Toussaint.
Uniaxial: ○ Gold and Krausz, △ Gold, □ Michel and Paradis.

the axes are stress and strain rate, and the solid circular symbols represent the relation between $P/\phi Dt$ and $U/\phi\psi D$ in Frederking and Gold's tests. It can be seen that they lie close to a line with slope 1/3; this is consistent with the prediction of reference-stress theory that in the power-law creep regime the ice force should be proportional to the one-third power of the velocity (since the reference stress $P/\phi Dt$ is then proportional to the one-third power of the strain rate $U/\phi\psi D$), and is also consistent with Frederking and Gold's observation that for fixed D/t force is proportional to the 0.34 power of velocity. Michel and Toussaint (1977) carried out another series of indentation tests on fresh-water ice at $-10°C$. The indentation velocity U varied from 5.84×10^{-7} m/s to 4.65×10^{-3} m/s, and the D/t ratio from 0.25 to 4. In Fig. 3, the solid triangular symbols represent the relation between the maximum values of $P/\phi Dt$ and $U/\phi\psi D$, for each of their tests.

The next step is to compare the observed relation between $P/\phi Dt$ and $U/\phi\psi D$ in indentation tests with the relation between stress and strain rate observed in uniaxial tests. Most authors report the results of "strength" tests at constant strain rate rather than creep tests at constant stress, but in a test at constant strain rate the sample will creep at the rate at which its deformation matches the deformation imposed by the machine, and in that sense a distinction between creep tests and strength tests for a creeping material is an artificial one. It follows that a plot of strength against strain rate in strength tests should be the same as a plot of stress against strain rate in creep tests. Gold (1977) pointed out that this was observed experimentally. The open symbols in Fig. 3 are data points for the relation between stress- and strain-rate in uniaxial compression, from Gold and Krausz (1971), Michel and Paradis (1976) and Gold (1977). All these tests were on columnar-grained S2 ice. The data reported by Michel and Paradis were measured at $-10°C$, at the same temperature and with the same type of ice used in the indentation experiments by Michel and Toussaint. Gold and Krausz' tests were on St. Lawrence River ice at $-9.5°C$.

If the theoretical model were exact, and the idealization were a complete description of the material behaviour, the solid symbols representing the indentation tests ought to lie on the same curve as the open symbols representing the creep data. It can be seen that the observations are consistent with this. The agreement is best at low strain rates. At straight rates above 3×10^{-5} s^{-1}, the indentation tests tend to give lower stresses than the creep tests at the same strain rates. The most likely explanation for this is that since deformation under the indentor is non-uniform, that the most severely-strained sections of the ice have strain rates markedly higher than $U/\phi\psi D$. When these local strain rates extend about 10^{-4} s^{-1}, the ice begings to fracture locally, and this effect modifies the stress field and lowers the load. Spalling and radial cracking will also be present.

The agreement turns out to extend to surprisingly large strain rates, beyond the level at which fracture becomes the dominant mechanism. This is consistent with the observation of Michel and Toussaint (1977) and Ralston (1979) that one can construct a "universal" curve, covering both the creep and fracture regime, for indentation and creep data.

PRIMARY CREEP

The load predicted by eqn. (20) is not immediately felt by the structure. On initial contact the ice sheet responds elastically, and then primary creep strains develop and stress redistribution takes place before the steady state is reached. In ice, primary creep strains are much greater than the initial elastic strains at a given level of stress, so it is the accumulation of primary creep strain in the structure that determines the rate of increase of load. The reference-stress technique can be extended to include primary creep through the assumption of time hardening. This leads to a prediction of the load after a time τ given by

$$P = \phi Dt\, F_\sigma \left[\frac{U/D}{\phi\psi}, \tau\right] \tag{24}$$

where $F_\sigma[x,\tau]$ is the uniaxial stress after a time t in a test conducted at constant strain rate x.

Ashby and Duval (1982) have suggested that at a constant strain rate $\dot{\epsilon}_a$ there is a unique relationship between the dimensionless variables

$$\tilde{\sigma} = \sigma/\sigma_{ss}\,, \tag{25}$$

and

$$\tilde{\tau} = \tau\dot{\epsilon}_a E/\sigma_{ss}\,, \tag{26}$$

where σ_{ss} is the steady-state stress corresponding to $\dot{\epsilon}_a$

$$\sigma_{ss} = \sigma_0(\dot{\epsilon}_a/\dot{\epsilon}_0)^{1/3}\,. \tag{27}$$

If time hardening is assumed then the relationship is

$$1 = B\tilde{\tau}^{-2/3}\tilde{\sigma}^{5/3}\exp(-(C\tilde{\tau}\tilde{\sigma}^2)^{1/3}) + \tilde{\sigma}^3, \tag{28}$$

where B is 37 and C is 1.6×10^{-2} for pure ice. Equation (2) is plotted in Fig. 4. Equation (1) suggests that this relationship is valid for the indentation problem when the dimensionless variables are defined as

$$\tilde{\sigma} = P/P_{ss} \,, \tag{29}$$

and

$$\tilde{\tau} = Ut\tau E/P_{ss}\psi \,. \tag{30}$$

The load is 80% of the steady-state value when $\tilde{\tau} = 85$.

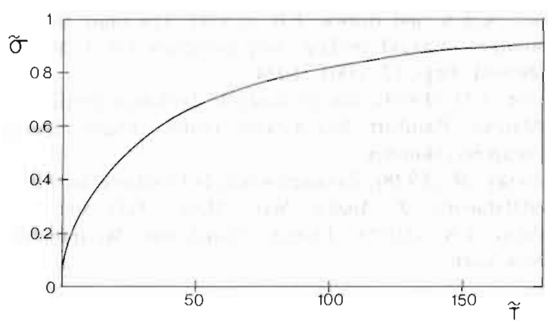

Fig. 4. Relation between dimensionless time and stress ratio.

CONCLUSION

Reference-stress theory provides a rational and straight-forward method for the calculation of ice forces in the creep mode. The method can be used to predict ice forces directly from measurements of the stress strain-rate relation, and is not limited to any particular functional relationship between stress and strain rate. Comparison with measurements in the laboratory encourages confidence in the application of the theory. The next step is to test the theory against large-scale tests and measurements on full-scale structures.

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APPENDIX: THE ELASTIC SOLUTION TO THE EDGE-INDENTATION PROBLEM

Texts on elasticity report a solution to the two-dimensional problem of normal indentation of an elastic half-space by a rigid plane indentor (Fig. 1). Sneddon (1951) gives the force corresponding to an indentor displacement Δ as $E\Delta\pi/2(1-\nu^2)\log 2$. However, the displacement field corresponding to this solution increases indefinitely with distance from the indentor, because of the presence of logarithmic terms: the displacement component u in the x-direction on the surface $x = 0$, for instance, is

$$u = \begin{cases} \Delta & \text{in } y < D/2 \,, \\ \Delta \left\{ 1 - \dfrac{\log(8y^2/D^2 - 1)}{2 \log 2} \right\} & \text{in } y > D/2 \,. \end{cases} \tag{A.1}$$

It follows that the relative displacement between the indentor and the far field is infinite. Hertz recognized this problem (Johnson, 1982), and he and Sadowsky both saw that it raised difficulties in applications.

A simple analysis emphasises the qualitative difference between the elastic (n=1) and creep ($n > 1$) indentation problems in two dimensions. In the far field, at a distance r (large by comparison with the indentor contact diameter) each stress component must be proportional to r^{-1}. It follows from (3) that each strain-rate component must be proportional to r^{-n}, and therefore that the relative velocity between the indentor and a point at a large distance R from the indentor must include a term of the form $\int^R r^{-n} dr$. If n is greater than 1, the integral behaves quite satisfactorily, and remains finite as R tends to infinity. If n equals 1, however, the integral includes a log R term, and diverges as R tends to infinity.

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