Ming Yuan He[1] *and John W. Hutchinson*[2]

Bounds for Fully Plastic Crack Problems for Infinite Bodies

REFERENCE: He, M. Y. and Hutchinson, J. W., **"Bounds for Fully Plastic Crack Problems for Infinite Bodies,"** *Elastic-Plastic Fracture, Volume I—Inelastic Crack Analysis, ASTM STP 803*, C. F. Shih and J. P. Gudas, Eds., American Society for Testing and Materials, 1983, pp. I-277–I-290.

ABSTRACT: For cracks in infinite bodies it is shown that modified principles of complementary potential energy and potential energy can be used to generate upper and lower bounds to the J-integral of the deformation theory of plasticity. These principles are used to obtain relatively tight numerical bounds on J for two basic plane strain problems: the finite crack in an infinite plane and the edge-crack in a semi-infinite plane. In both problems the material is incompressible with a pure power relation between stress and strain. Upper bounds for the plane stress problems are also given.

KEY WORDS: nonlinear fracture mechanics, fully plastic crack problems, J-integral, bounds, elastic-plastic fracture

In the first section of this paper we derive principles for obtaining upper and lower bounds for the J-integral in certain infinite or semi-infinite crack problems in which the only length quantity involved is the crack length itself. Under these circumstances J, the energy release rate per unit of crack extension in a deformation theory material, is simply related to the minimum of the potential energy functional and the minimum of the complementary energy functional, each of which is modified appropriately for an unbounded region. Any statically admissible stress field in conjunction with the modified complementary energy functional generates an upper bound to J, while any kinematically admissible displacement field used with the modified potential energy functional gives a lower bound.

The principles are then implemented to obtain numerical bounds for two basic fully plastic problems in plane strain: the finite crack of length $2a$ in the

[1]Researcher, Institute of Mechanics, Chinese Academy of Sciences, Beijing, China.
[2]Professor of applied mechanics, Division of Applied Sciences, Harvard University, Cambridge, Mass. 02138.

infinite plane and the crack of length a normal to the edge of a semi-infinite plane. Numerical results for the upper bound to J are also given for the corresponding problems under plane-stress conditions.

The numerical results are obtained for the case of a small strain incompressible solid characterized in simple tension by

$$\epsilon/\epsilon_0 = \alpha(\sigma/\sigma_0)^n \tag{1}$$

where ϵ_0 and σ_0 are a reference strain and stress and α is an extra constant introduced for convenience of application. The tensile relation is generalized to multi-axial states by J_2 deformation theory according to

$$\frac{\epsilon_{ij}}{\epsilon_0} = \frac{3}{2}\alpha\left(\frac{\sigma_e}{\sigma_o}\right)^{n-1}\frac{s_{ij}}{\sigma_0} \tag{2}$$

where s_{ij} is the stress deviator and σ_e is the effective stress defined by

$$\sigma_e = (\tfrac{3}{2}\, s_{ij} s_{ij})^{1/2} \tag{3}$$

With an effective strain defined by

$$\epsilon_e = (\tfrac{2}{3}\, \epsilon_{ij}\, \epsilon_{ij})^{1/2} \tag{4}$$

σ_e and ϵ_e satisfy the tensile relation, Eq 1.

The numerical results are obtained using a Rayleigh-Ritz procedure to minimize the respective functionals with respect to a family of kinematically or statically admissible fields with free amplitude factors. This same numerical procedure was employed in an earlier paper [*1*][3] to bound J from below for the finite crack in the infinite plane and for a penny-shaped crack in an infinite body. In still earlier work, Ranaweera and Leckie [*2*] used finite-element procedures based on both displacements and stresses together with the corresponding two unmodified minimum principles to evaluate J for several configurations of finite extent. Their results do appear to have a bound-like character even though the bounding principles do not strictly apply to finite bodies.

Upper and Lower Bounds to J

Attention is restricted to a small strain deformation theory of plasticity for homogeneous bodies in which the strain energy density and complementary stress energy density are

$$W(\epsilon) = \int_0^{\epsilon} \sigma_{ij}\, d\epsilon_{ij} \quad \text{and} \quad U(\sigma) = \int_0^{\sigma} \epsilon_{ij}\, d\sigma_{ij} \tag{5}$$

[3]The italic numbers in brackets refer to the list of references appended to this paper.

To construct the minimum principles for an infinite or semi-infinite body loaded at infinity, let σ^∞, ϵ^∞, and $\mathbf{u}^\infty$ denote the stresses, strains, and displacements associated with the uniform field of the loaded body in the absence of the crack. Denote additional quantities by a tilde so that the total quantities in the presence of a crack are given by

$$\sigma = \sigma^\infty + \tilde{\sigma}, \qquad \epsilon = \epsilon^\infty + \tilde{\epsilon}, \qquad \mathbf{u} = \mathbf{u}^\infty + \tilde{\mathbf{u}} \tag{6}$$

Consider a family of finite bodies with an outer boundary S_R of radius R on which tractions $T_i = \sigma_{ij}{}^\infty n_j$ are prescribed as shown in Fig. 1. The traction-free crack has length (or half-length) a and for the edge-crack problem the edge ($x_1 = 0$) is also traction-free. The remote stress σ^∞ must be consistent with the traction-free conditions on boundaries other than the crack. For example, $\sigma_{ij}{}^\infty n_j$ must vanish on the edge $x_1 = 0$ for the edge-crack problem.

The potential and complementary potential energy functionals for the finite body are

$$PE = \int_{V_R} W(\epsilon)dV - \int_{S_R} \sigma_{ij}{}^\infty n_j u_i dS \tag{7}$$

and

$$CE = \int_{V_R} U(\sigma)dV \tag{8}$$

where V_R denotes the volume per unit thickness exterior to the crack and interior to S_R. The stress field in Eq 8 must satisfy equilibrium and the traction-free boundary conditions on the crack and on any other traction-free

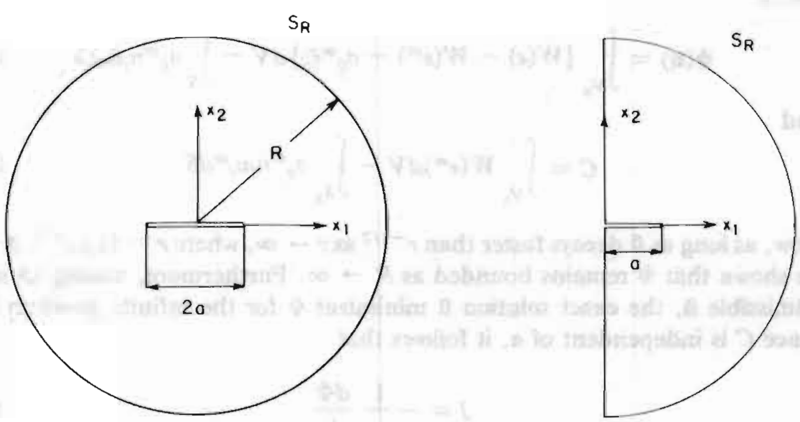

FIG. 1—*Crack geometry.*

boundary. It must satisfy the traction condition $\sigma_{ij}n_j = \sigma_{ij}^{\infty}n_j$ on S_R. The strain in Eq 7 is derived from the displacement field using the linear strain-displacement equations. Subject to standard convexity conditions on the energy density functions, which will be assumed, the two functionals are minimized by the respective exact fields. For the exact solution it is easily shown that

$$CE = -PE \tag{9}$$

and J, defined as the energy release rate with $\sigma_{ij}^{\infty}n_j$ held constant on S_R, is

$$J = -\frac{1}{m}\frac{dPE}{da} = \frac{1}{m}\frac{dCE}{da} \tag{10}$$

Here $m = 1$ when one crack tip is present, as in the edge-crack problem, and $m = 2$ when the crack has two tips.

We now manipulate Eqs 7 and 8 into forms such that the contributions which depend on a remain bounded as R becomes infinite. First we deal with Eq 7. By the principle of virtual work

$$\int_{S_R} \sigma_{ij}^{\infty}n_j\tilde{u}_i dS = \int_{V_R} \sigma_{ij}^{\infty}\tilde{\epsilon}_{ij}dV_R + \int_S \sigma_{ij}^{\infty}n_j\tilde{u}_i dS \tag{11}$$

where S is the surface of the crack and $\mathbf{n}$ is the unit normal to S pointing into V_R. Equation 7 can be written as

$$PE = \Phi + C \tag{12}$$

where

$$\Phi(\tilde{\mathbf{u}}) = \int_{V_R} \{W(\epsilon) - W(\epsilon^{\infty}) - \sigma_{ij}^{\infty}\tilde{\epsilon}_{ij}\}dV - \int_S \sigma_{ij}^{\infty}n_j\tilde{u}_i dS \tag{13}$$

and

$$C = \int_{V_R} W(\epsilon^{\infty})dV - \int_{S_R} \sigma_{ij}^{\infty}n_j u_i^{\infty}dS \tag{14}$$

Now, as long as $\tilde{\mathbf{u}}$ decays faster than $r^{-1/2}$ as $r \to \infty$, where $r = (x_ix_i)^{1/2}$, it can be shown that Φ remains bounded as $R \to \infty$. Furthermore, among all such admissible $\tilde{\mathbf{u}}$, the exact solution $\tilde{\mathbf{u}}$ minimizes Φ for the infinite problem [3]. Since C is independent of a, it follows that

$$J = -\frac{1}{m}\frac{d\Phi}{da} \tag{15}$$

and this relation is valid in the limit as R becomes unbounded.

For either the infinite plane with crack of length $2a$ or the semi-infinite plane with the edge-crack of length a, the crack length is the only length quantity and thus Φ must have the form

$$\Phi = a^2 f(\sigma^\infty) \tag{16}$$

where f depends implicitly on the material properties and on σ^∞ but not on a. Thus, from Eqs 15 and 16

$$J = -\frac{2}{ma}\,\Phi_{\min} \tag{17}$$

where the notation $\Phi_{\min}$ is used to emphasize that Φ is evaluated using the exact solution. Any estimate of J obtained from

$$J = -\frac{2}{ma}\,\Phi(\tilde{u}) \tag{18}$$

using an admissible additional field $\tilde{\mathbf{u}}$ necessarily leads to a *lower bound* to J since $\Phi(\tilde{\mathbf{u}}) \geq \Phi_{\min}$.

Next, we rewrite Eq 8 as

$$CE = F + C' \tag{19}$$

where

$$F(\tilde{\sigma}) = \int_{V_R} \{U(\sigma) - U(\sigma^\infty) - \tilde{\sigma}_{ij}\epsilon_{ij}^{\infty}\}\, dV \tag{20}$$

and

$$C' = \int_{V_R} U(\sigma^\infty) dV \tag{21}$$

Here we have made use of the fact that

$$\int_{V_R} \tilde{\sigma}_{ij}\epsilon_{ij}^{\infty} dV = 0$$

since $\tilde{\sigma}_{ij}n_j = 0$ on S_R and $\tilde{\sigma}_{ij}n_j u_i^\infty = -\sigma_{ij}^\infty n_j u_i^\infty$ on the crack faces, which integrates to zero. If the additional stresses $\tilde{\sigma}$ decay faster than $r^{-3/2}$ as $r \to \infty$, then Eq 20 is bounded in the infinite problem. Since C' is independent of a

$$J = \frac{1}{m}\frac{dF}{da} \tag{22}$$

Furthermore, among all statically admissible $\tilde{\sigma}$ the exact solution minimizes F, as can be shown using an approach similar to that employed in Ref *3* for the potential energy formulation. From dimensional considerations it follows that F is proportional to a^2 for either of the plane problems so that

$$J = \frac{2}{ma} F_{\min} \tag{23}$$

By virtue of the minimizing role of the exact solution, any estimate of J obtained from

$$J = \frac{2}{ma} F(\tilde{\sigma}) \tag{24}$$

using a statically admissible additional stress field gives an *upper bound*.

Numerical Method

For the upper bounds a stress function was used to generate equilibrium additional stress fields according to

$$\tilde{\sigma}_{11} = \chi_{,22}, \qquad \tilde{\sigma}_{22} = \chi_{,11}, \qquad \text{and} \quad \tilde{\sigma}_{12} = -\chi_{,12} \tag{25}$$

The power-law material, Eq 2, is inherently incompressible and thus in the lower-bound calculations a stream function could be used to give the additional displacements as

$$\tilde{u}_1 = \psi_{,2} \quad \text{and} \quad \tilde{u}_2 = -\psi_{,1} \tag{26}$$

In each case, the function was represented as a linear sum of admissible functions in the form

$$\chi = \sum_{i=1}^{K} A_i \chi^{(i)} \quad \text{or} \quad \psi = \sum_{i=1}^{K} A_i \psi^{(i)} \tag{27}$$

where the A_i's are free variables chosen to minimize F or Φ. This minimization was achieved by a Newton-Raphson method. With the upper-bound formulation to illustrate the method, the minimum condition is

$$\frac{\partial F}{\partial A_k} = 0 \qquad k = 1, K \tag{28}$$

where

$$\frac{\partial F}{\partial A_k} = \int_V (\epsilon_{ij} - \epsilon_{ij}^{\infty}) s_{ij}^{(k)} dV \tag{29}$$

and $\mathbf{s}^{(k)}$ is derived from $\chi^{(k)}$ so that

$$s_{ij} = \sum_{k=1}^{K} A_k s_{ij}^{(k)} \tag{30}$$

If $\{A_j\}$ is an estimate of the solution to Eq 28, the improved estimate $\{A_j + \Delta A_j\}$ from the next iteration is obtained from

$$\sum_{m=1}^{K} \Delta A_m \frac{\partial^2 F}{\partial A_m \partial A_p} = -\frac{\partial F}{\partial A_p} \tag{31}$$

where

$$\frac{\partial^2 F}{\partial A_m \partial A_p} = \int_V \frac{3}{2} \sigma_e^{n-1} \left[s_{ij}^{(m)} s_{ij}^{(p)} + \frac{3}{2} \frac{(n-1)}{\sigma_e^2} (s_{ij} s_{ij}^{(m)})(s_{kl} s_{kl}^{(p)}) \right] dV \tag{32}$$

Corresponding expressions for the lower bound based on Φ and the first of Eq 27 are readily derived.

The integrals over the body in Eqs 20, 29, and 32 were evaluated using a mapping technique together with numerical integration in the mapping plane. With reference to Fig. 2, the physical plane in both plane problems investigated here was mapped into the unit circle in the ζ-plane using the conformal map

$$z = \omega(\zeta) \equiv \frac{a}{2} (\zeta + \zeta^{-1}) \tag{33}$$

where $z = x_1 + ix_2$ and $\zeta = \xi + i\eta = \mu i e^{-i\phi}$. The planar-polar coordinates μ and ϕ in the mapping plane were taken as the independent variables in the functional representation of χ and ψ, as described in the following. Integrations over the body were performed using iterated 10-point Gaussian quadrature formulas for μ ranging from 0 to 1 and ϕ ranging from θ to $\pi/2$. This same technique was used in Refs *1* and *4*, where it is described in more detail. Test cases based on known linear solutions and on special trial functions for the nonlinear material indicated that the numerical integration of F or Φ is accurate to at least three or four significant figures.

The numerical method in the present study differed from that in our previous paper [*1*] in that here explicit expressions for the partial derivatives of Φ and F with respect to the amplitude factors (for example, Eqs 29 and 32) were used. In the earlier work these derivatives were evaluated directly from the functional by numerical differentiation. The present procedure resulted in a considerable reduction in computation time and led to better numerical conditioning in some instances.

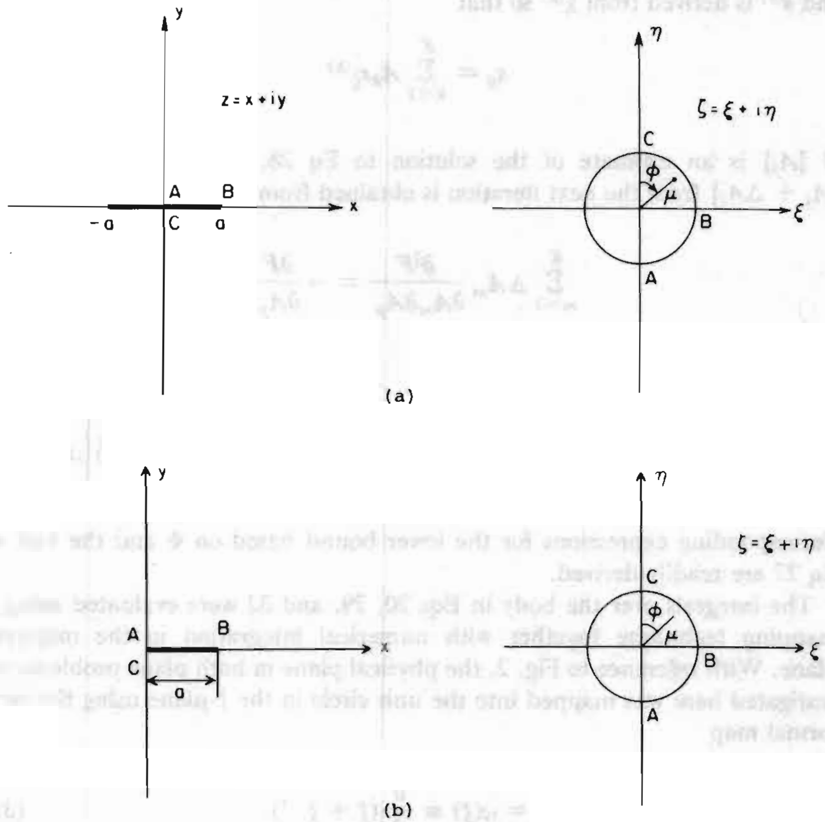

FIG. 2—*Mapping from physical plane to mapping plane:* (a) *central crack;* (b) *edge-crack.*

Bounds for the Crack in the Infinite Plane

As indicated in Fig. 1, the crack of length $2a$ lies on the x_1-axis, the one nonzero remote in-plane stress component is $\sigma_{22}{}^{\infty} = \sigma^{\infty}$, and plane-strain conditions ($\epsilon_{33} = 0$) are assumed. The form of the solution for J for the pure power-hardening material, Eq 2, is [1]

$$J = a\sigma_e{}^{\infty}\,\epsilon_e{}^{\infty}h(n) \tag{34}$$

where, from Eqs 3 and 4

$$\sigma_e{}^{\infty} = \frac{\sqrt{3}}{2}\,|\sigma^{\infty}| \quad \text{and} \quad \epsilon_e{}^{\infty} = \frac{2}{\sqrt{3}}\,|\epsilon_{22}{}^{\infty}| = \alpha\epsilon_0\left(\frac{\sigma_e{}^{\infty}}{\sigma_0}\right)^n \tag{35}$$

The bounds given in the following are for $h(n)$.

For the *lower bound* the stream function, Eq 26, was taken as

$$\psi = \sum_{k=1}^{N} \sum_{j=1}^{M} A_{kj} \left[\mu^{j-1} \sin 2k\phi + (-1)^{k+1} 2k(\phi - \pi/2)\right] \tag{36}$$

which is the same representation used in Ref *1*, except that more terms were used to obtain the present results. This choice is consistent with the symmetry of the solution with respect to the two Cartesian axes and, as discussed in Ref *1*, gives rise to a crack opening of the approximate form $f(x_1)\sqrt{a^2 - x_1^2}$ where $f(a)$ is bounded. Expressions for partial derivatives of ψ (and χ) with respect to Cartesian coordinates, which are needed in the evaluation of Φ, are obtained in terms of the derivatives with respect to μ and ϕ with the aid of standard change of variable formulas as illustrated in Refs *1* and *4*.

The stress function for the additional stresses Eq 25 for the *upper bound* was taken to be

$$\chi = \frac{\sigma^{\infty}}{2} (\mu + \mu^{-1})^2 [(1 - \mu^2)^s - 1] \sin^2\phi +$$

$$- (1 + b_i) f_p(\phi)] + \sum_{k=1}^{N} \sum_{j=1}^{M} A_{kj} (1 - \mu)^{3 - [2/(n+1)] + j} \cos 2k\phi \tag{37}$$

This choice is also consistent with the twofold symmetry and each individual contribution gives rise to additional stress components which decay faster than $r^{-3/2}$ for large r. The first term in Eq 37 gives $\bar{\sigma}_{12} = 0$ and $\bar{\sigma}_{22} = -\sigma^{\infty}$ on the crack faces, while each of the other terms makes no traction contribution. Thus, from Eq 6, the total traction vanishes on the crack faces. By numerical experimentation it was found that a relatively large value of the exponent s in the first term was best and the choice $s = 7$ was made. The individual terms in the double sum in Eq 37 with $j = 1$ each has a stress singularity at the tip of the order $r^{-1/(n+1)}$, which is in agreement with the order of the actual singularity. The terms in the double sum make a contribution to σ_{11} which is zero on the crack faces. The second set of terms in Eq 37 is capable of representing an arbitrary distribution of σ_{11} along the crack faces. The functions $f_i(\phi)$ form a complete set with the required symmetry conditions $f = f' = 0$ at $\phi = 0$ and $\pi/2$. They are given by

$$f_i(\phi) = \cos \lambda_i(\phi - \pi/2) + b_i \cosh \lambda_i(\phi - \pi/2) \tag{38}$$

where λ_i is the root of (ordered in ascending magnitude with $\lambda_1 = 1.5056187$)

$$\cos(\lambda_i \pi/2) \sinh(\lambda_i \pi/2) + \sin(\lambda_i \pi/2) \cosh(\lambda_i \pi/2) = 0 \tag{39}$$

and

$$b_i = \sin(\lambda_i \pi/2)/\sinh(\lambda_i \pi/2) \tag{40}$$

The computed bounds to $h(n)$ as defined in Eq 34 are given in Table 1 for values of n between 1 and 7. For $n \leq 3$ the spread between the bounds is less than 2 percent. For $n = 5$ it is 4 percent and for $n = 7$ it is 7 percent. The lower bound was computed with $N = 5$ and $M = 6$ for a total of 30 free amplitudes.[4] The upper bound was determined using 34 free amplitudes with $N = 5$, $M = 6$, and $P = 5$.

Included in Table 1 and plotted in Fig. 3 are numerical values based on the

TABLE 1—*Upper- and lower-bound estimates of* h *for plane-strain cracks in an infinite plane.*

	$n = 1.0$	$n = 1.5$	$n = 2.0$	$n = 3.0$	$n = 4.0$	$n = 5.0$	$n = 7.0$
Upper bound	3.156	3.884	4.517	5.616	6.579	7.456	9.052
Lower bound	3.141	3.856	4.470	5.511	6.390	7.152	8.421
$\pi\sqrt{n}$	3.141	3.848	4.443	5.441	6.282	7.024	8.312

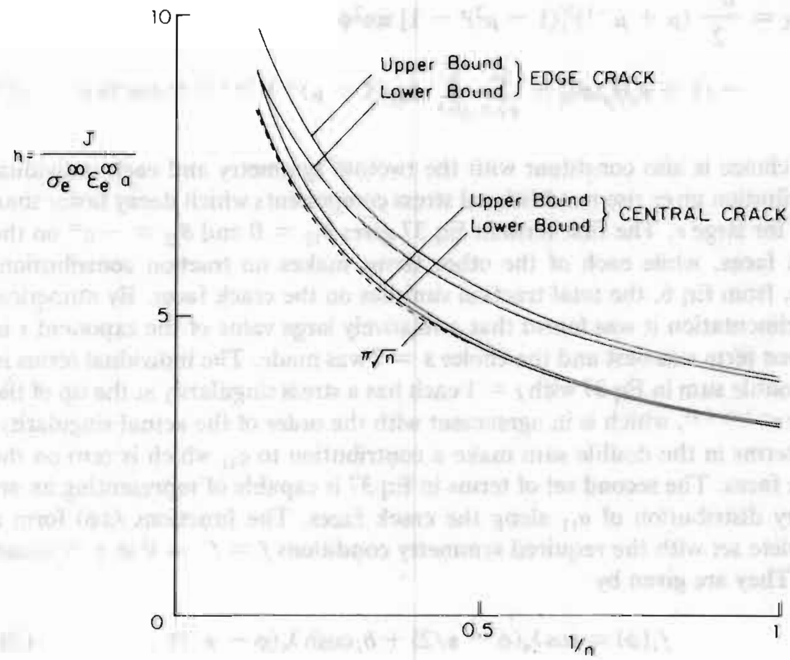

FIG. 3—*Bounds for the central crack in the infinite body and the edge crack in the semi-infinite plane under plane-strain conditions.*

[4]The results reported in Ref *1* were computed with $N = 3$ and $M = 4$; for $n = 5$ the present results are about 5 percent higher. This reflects the convergence rate with an increasing number of terms.

approximate result $h(n) = \pi\sqrt{n}$ obtained in Ref *1*. It is seen that this simple approximation is only slightly below the lower bound in the range $1 < n \leq 7$ and is within 3 percent of the upper bound for $n \leq 3$ and within 8 percent for $n = 7$. As described in Ref *1* this result is a special case of a formula derived under more general remote stress conditions where

$$\sigma_{22}^{\infty} = S, \qquad \sigma_{11}^{\infty} = T, \qquad \text{and} \quad \sigma_e^{\infty} = \frac{\sqrt{3}}{2}|S - T| \tag{41}$$

That formula is

$$J = \frac{3\pi\sqrt{n}}{4} a\sigma_e^{\infty}\epsilon_e^{\infty}\left(\frac{S}{\sigma_e^{\infty}}\right)^2 \tag{42}$$

For $n = 1$ the formula is exact for all S and T, while for $n \neq 1$ the formula has an error of order $(S/\sigma_e^{\infty})^4$ when S/σ_e^{∞} is regarded as small. The present bounds indicate that the formula retains its accuracy (for most purposes) for S/σ_e^{∞} at least as large as $2/\sqrt{3}$, corresponding to remote plane-strain tension. The reader is referred to the discussion in Ref *1* for an explanation of the unbounded character of the results in the rigid-perfectly plastic limit as $n \to \infty$.

Bounds for the Edge-Crack in the Semi-Infinite Plane

Here the crack of length a is again under plane-strain conditions with a remote stress $\sigma_{22}^{\infty} = \sigma^{\infty}$ so that Eqs 34 and 35 still hold.

For the *lower bound* the stream function was taken as

$$\psi = (\phi - \pi/2)\left[A_0(1-\mu) + \sum_{i=1}^{P} A_i(\mu^{i+1} - \mu)\right] + \sum_{k=1}^{N}\sum_{j=1}^{M} A_{kj}\left[\mu^{j-1}\sin 2k\phi + 2k(-1)^{k+1}(\phi - \pi/2)\mu\right] \tag{43}$$

This choice, like the previous one for the crack in the infinite plane, has a crack opening of the form $f(x_1)\sqrt{a - x_1}$ with associated strains and stresses which are less singular than the actual solution when $n > 1$. The representation for the stress function χ used to calculate the *upper bound* is the same as Eq 37, except that the double sum is replaced by

$$\sum_{k=1}^{N}\sum_{j=1}^{M} A_{kj}(1-\mu)^{3-[2/(n+1)]+j} f_k(\phi) \tag{44}$$

where $f_k(\phi)$ is defined by Eq 38. Each term in the representation for χ satisfies the zero traction condition on the edge of the plane $x_1 = 0$. The total stress distribution meets the zero traction condition on the crack faces. Thus, χ produces a statically admissible stress distribution for any set of amplitudes.

Computed bounds to $h(n)$ in Eq 34 are presented in Table 2 and plotted in Fig. 3. The lower bound was determined using 34 free parameters ($N = 6$, $M = 5$, $P = 3$), and the upper bound was computed with a total of 29 free parameters ($N = 5$, $M = 5$, $P = 5$). Included in Table 2 are values of the normalized crack mouth opening, $\delta = 2u_2(0, 0^+)$, at the edge of the plane where

$$g(n) = \frac{\delta}{a\epsilon_e^\infty} \tag{45}$$

These values were obtained from the stream function computation.

No comparably simple formula to Eq 42 is available for the edge crack. For $n = 1$ at the same remote state of uniaxial tension σ^∞, $h = (1.1215)^2\pi$ for the edge crack while $h = \pi$ for the crack in the infinite plane [5]. In Table 2 we have included the values from $(1.1215)^2\pi\sqrt{n}$, which would be a good approxi-

TABLE 2—*Upper- and lower-bound estimates of* h *and* g *for plane-strain edge cracks.*

		$n = 1.0$	$n = 1.5$	$n = 2.0$	$n = 3.0$	$n = 4.0$	$n = 5.0$	$n = 7.0$
h	Upper bound	3.952	4.771	5.448	6.568	7.507	8.336	9.792
	Lower bound	3.871	4.639	5.264	6.281	7.108	7.812	8.972
	$(1.1215)^2\pi\sqrt{n}$	3.951	4.844	5.593	6.853	7.909	8.843	10.46
g		4.726	5.271	5.731	6.467	7.080	7.613	8.486

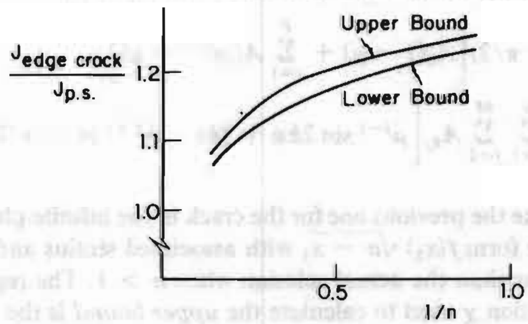

FIG. 4—*Ratio of bounds to* J *for edge crack to estimate of* J *for central crack as a function of* $1/n$.

TABLE 3—*Upper-bound estimates of* h *for plane-stress cracks.*

	$n = 1.0$	$n = 1.5$	$n = 2.0$	$n = 3.0$	$n = 4.0$	$n = 5.0$	$n = 7.0$
Edge crack	3.955	4.874	5.656	6.985	8.119	9.124	10.88
Central crack	3.190	3.914	4.561	5.723	6.774	7.751	9.560

mation if the ratio of J for the edge crack to J for the crack in the infinite plane were independent of n. This is clearly not the case as can also be seen from the plot of this ratio in Fig. 4. The curves of Fig. 4 suggest that the ratio of the two J-values may approach unity for large n.

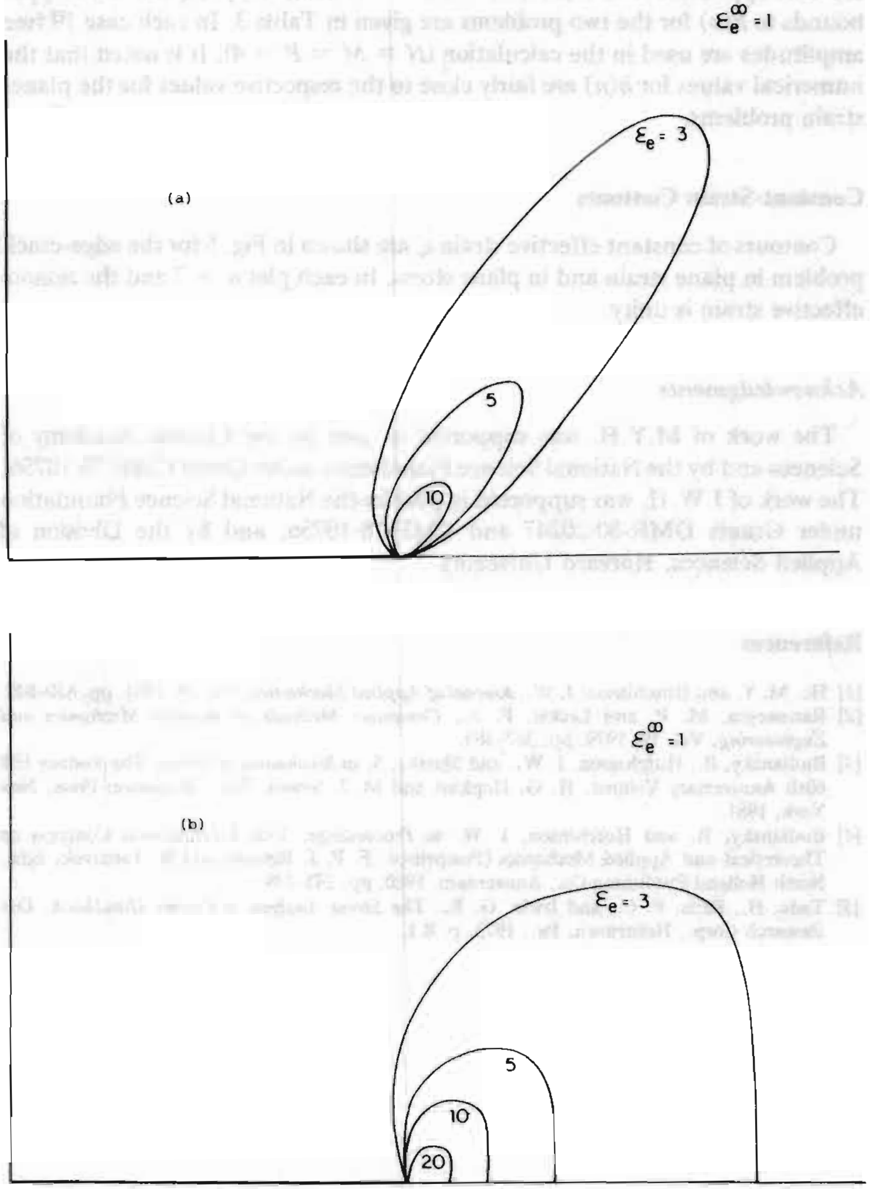

FIG. 5—*Contours of constant effective strain with the remote effective strain as unity and* n = 7: (a) *edge crack in plane strain;* (b) *edge crack in plane stress.*

Upper Bounds for the Plane-Stress Problems

The method outlined in the preceding for determining the upper bound to J in plane strain applies to plane stress with the minor modification that $\sigma_{33} = 0$ and ϵ_{33} is not required to be zero. The representations for the stress functions are still applicable and $h(n)$ continues to be defined by Eq 34. The upper bounds to $h(n)$ for the two problems are given in Table 3. In each case 19 free amplitudes are used in the calculation ($N = M = P = 4$). It is noted that the numerical values for $h(n)$ are fairly close to the respective values for the plane-strain problems.

Constant-Strain Contours

Contours of constant effective strain ϵ_e are shown in Fig. 5 for the edge-crack problem in plane strain and in plane stress. In each plot $n = 7$ and the remote effective strain is unity.

Acknowledgments

The work of M.Y.H. was supported in part by the Chinese Academy of Sciences and by the National Science Foundation under Grant CME-78-10756. The work of J.W.H. was supported in part by the National Science Foundation under Grants DMR-80-20247 and CME-78-10756, and by the Division of Applied Sciences, Harvard University.

References

[1] He, M. Y. and Hutchinson, J. W., *Journal of Applied Mechanics*, Vol. 48, 1981, pp. 830–840.
[2] Ranaweera, M. P. and Leckie, F. A., *Computer Methods in Applied Mechanics and Engineering*, Vol. 19, 1979, pp. 367–389.
[3] Budiansky, B., Hutchinson, J. W., and Slutsky, S. in *Mechanics of Solids*, The Rodney Hill 60th Anniversary Volume, H. G. Hopkins and M. J. Sewell, Eds., Pergamon Press, New York, 1981.
[4] Budiansky, B. and Hutchinson, J. W. in *Proceedings*, 15th International Congress on Theoretical and Applied Mechanics (Postprints), F. P. J. Rimrott and B. Tabarrok, Eds., North-Holland Publishing Co., Amsterdam, 1980, pp. 243–249.
[5] Tada, H., Paris, P. C., and Irwin, G. R., *The Stress Analysis of Cracks Handbook*, Del. Research Corp., Hellertown, Pa., 1973, p. 8.1.