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CONSTITUTIVE POTENTIALS FOR DILUTELY VOIDED NONLINEAR MATERIALS *

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Constitutive relations are derived for an incompressible, isotropic power-law matrix material containing a dilute concentration of spherical voids. The derivation is made for a nonlinearly viscous material used to characterize steady creep. However, the theory applies equally well to small strain nonlinear elasticity (deformation theory), and an extension to a rate-independent flow theory is also discussed. The starting point and key element in the formulation is the potential function for an isolated spherical void in an infinite block of power-law material. Approximate, but accurate, representations for this potential function are given. The overall constitutive relation governing the behavior of the dilutely voided solid is obtained simply and directly using the void potential. An assessment of the range of validity of the dilute concentration results is obtained using numerical solutions to the problem of a spherical void centered in a sphere of finite radius made of the power-law material. The potential function is also given for a dilute concentration of aligned penny-shaped cracks in the same power-law material.

1. Synopsis of results

The matrix material is an incompressible nonlinearly viscous material frequently used as a prototype to characterize steady creep. In simple tension the strain-rate and stress are related by the power-law formula

$$\dot{\epsilon} = \alpha(\sigma/\sigma_0)^n, \tag{1.1}$$

where α and σ_0 are a reference strain-rate and stress. For multiaxial states of stress define a potential of the stress by

$$\phi(\boldsymbol{\sigma}) = (\alpha\sigma_0/(n+1))(\sigma_e/\sigma_0)^{n+1} \tag{1.2}$$

so that the strain-rate is

$$\dot{\epsilon}_{ij} = \partial\phi/\partial\sigma_{ij} = \tfrac{3}{2}\alpha(\sigma_e/\sigma_0)^{n-1}s_{ij}/\sigma_0, \tag{1.3}$$

where $\boldsymbol{s}$ is the stress deviator and $\sigma_e = (\frac{3}{2}s_{ij}s_{ij})^{1/2}$ is the effective stress.

In Section 2 we consider a representative macroscopic block of material of volume V containing traction-free voids and subject to macroscopic, or overall, stress $\boldsymbol{\Sigma}$. The macroscopic potential is related to the distribution of the local potential by

$$\Phi(\boldsymbol{\Sigma}) = V^{-1}\int_{V_M}\phi\,\mathrm{d}V, \tag{1.4}$$

where V_M denotes the portions of V occupied by the matrix material. The macroscopic strain-rate is given by

$$\dot{E}_{ij} = \partial\Phi/\partial\Sigma_{ij}. \tag{1.5}$$

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For a dilute concentration of spherical voids with volume fraction ρ, the macroscopic potential is given by

$$\Phi = \phi(\boldsymbol{\Sigma}) + \rho \Phi_{\mathrm{v}}(\boldsymbol{\Sigma}). \tag{1.6}$$

Here Φ_{v} is an appropriately defined change in potential caused by the introduction of an isolated spherical void in an infinite block of the matrix material which is subject to remote stress $\boldsymbol{\Sigma}$.

The problem of the isolated spherical void in the infinite block of material is taken up in Section 3. By isotropy, Φ_{v} depends only on the three invariants of $\boldsymbol{\Sigma}$. Let the remote effective stress invariant be defined as

$$\Sigma_{\mathrm{e}} = \left(\tfrac{3}{2} S_{ij} S_{ij}\right)^{1/2}, \tag{1.7}$$

where $\boldsymbol{S}$ is the deviator of $\boldsymbol{\Sigma}$, and let the remote mean stress invariant be

$$\Sigma_{\mathrm{m}} = \tfrac{1}{3} \Sigma_{kk}. \tag{1.8}$$

We will argue that the dependence of Φ_{v} on the third invariant is relatively minor compared to its dependence on Σ_{e} and Σ_{m}, and the dependence on the third invariant will be neglected. Then, because Φ_{v} is homogeneous of degree $n+1$ in $\boldsymbol{\Sigma}$, one can write

$$\Phi_{\mathrm{v}} = \alpha \sigma_0 (\Sigma_{\mathrm{e}}/\sigma_0)^{n+1} f(X, n), \tag{1.9}$$

where

$$X = \Sigma_{\mathrm{m}}/\Sigma_{\mathrm{e}} \tag{1.10}$$

is the measure of stress triaxiality. Previously published results for an isolated spherical void, together with some newly reported results, are used to arrive at an approximation to $f(X, n)$ for n ranging from 1 to ∞.

By (1.5) and (1.6) the macroscopic strain-rate derived from (1.9) for the dilutely voided material is given in terms of the macroscopic stress $\boldsymbol{\Sigma}$ by

$$\dot{E}_{ij} = \tfrac{3}{2} \alpha \left(\frac{\Sigma_{\mathrm{e}}}{\sigma_0}\right)^{n-1} \frac{S_{ij}}{\sigma_0} + \alpha \rho \left\{ \frac{3}{2} \left(\frac{\Sigma_{\mathrm{e}}}{\sigma_0}\right)^{n-1} \frac{S_{ij}}{\sigma_0} \left((n+1) f - X \frac{\partial f}{\partial X} \right) + \frac{1}{3} \left(\frac{\Sigma_{\mathrm{e}}}{\sigma_0}\right)^{n} \frac{\partial f}{\partial X} \delta_{ij} \right\}, \tag{1.11}$$

where δ_{ij} is the Kronecker delta. Some implications of (1.11) are discussed in Section 4, where it is also discussed in relation to McMeeking's (1982) work on dilutely voided nonlinear materials. In particular, extension of the results to rate-independent plastic solids described by flow (incremental) theory will be discussed.

In Section 5, a sphere of the matrix material of finite radius with a centered spherical void is employed as a model macroscopic material element with a non-dilute void volume fraction ρ. Numerical results are presented for this problem for the case of axisymmetric uniform tractions derived from $\boldsymbol{\Sigma}$ and applied on the outer boundary. These results are used to gain some insight into the range of validity of the dilute approximation and to what extent this range is a function of the material nonlinearity.

Lastly, in Section 6, a potential is given for the power-law material weakened by a dilute concentration of penny-shaped cracks aligned in the principal axes of overall stress.

2. Potentials for voided nonlinearly viscous materials

Derivations of constitutive behavior of heterogeneous nonlinear materials based on the use of potentials have the advantage that one works with scalar, not tensor, functions. The advantages become particularly evident when approximations must be made, as will be illustrated below.

Following techniques developed by Hill (1967) for dealing with heterogeneous materials, we consider as a representative macroscopic volume element a block of material with volume V consisting of traction-free voids or micro-cracks embedded in an incompressible nonlinearly viscous matrix material. The matrix material is characterized by a potential of the stress, $\phi(\boldsymbol{\sigma})$, such that the strain-rate is given by

$$\dot{\epsilon}_{ij} = \partial\phi/\partial\sigma_{ij}. \tag{2.1}$$

The power-law potential (1.2) characterizes one such material. The region occupied by the matrix material is denoted by V_M, while the voids and cracks occupy the region of V denoted by V_v.

Throughout this article lower case symbols will be used for local stresses, strain-rates, velocities and potentials, which are functions of position in the matrix material. Upper case symbols will be used to represent the corresponding macroscopic quantities. Let A denote the outer surface of the block. Let uniform tractions $\Sigma_{ij}n_j$ be prescribed over A, where $\boldsymbol{n}$ is the outward unit normal to A and $\boldsymbol{\Sigma}$ is the macroscopic stress whose components do not vary over the surface of the macroscopic block of material. With $\boldsymbol{v}$ as the velocity resulting from the uniform tractions, the macroscopic strain-rate is defined in the standard way in terms of velocities on the surface of V by

$$\dot{E}_{ij} = \tfrac{1}{2}V^{-1}\int_A (v_i n_j + v_j n_i)\,\mathrm{d}A. \tag{2.2}$$

To derive the relation between the macroscopic and local potentials (Rice, 1970; Hutchinson, 1983), note that

$$\dot{E}_{ij}\,\mathrm{d}\Sigma_{ij} = V^{-1}\int_A v_i n_j\,\mathrm{d}\Sigma_{ij}\,\mathrm{d}A = V^{-1}\int_A v_i n_j\,\mathrm{d}\sigma_{ij}\,\mathrm{d}A = V^{-1}\int_{V_M} \dot{\epsilon}_{ij}\,\mathrm{d}\sigma_{ij}\,\mathrm{d}V = V^{-1}\int_{V_M} \mathrm{d}\phi\,\mathrm{d}V. \tag{2.3}$$

Here, use has been made of (i) the traction condition on A, (ii) equilibrium, (iii) the condition that the voids or cracks are traction-free, and (iv) equation (2.1). It follows from the fact that (2.3) holds for arbitrary $\mathrm{d}\boldsymbol{\Sigma}$ that macroscopic potential Φ is given by (1.4) and that the macroscopic strain-rate can be derived from it by (1.5).

Now let us restrict our attention to a block with a *dilute concentration* of voids with volume fraction $\rho = V_v/V$. If the field induced in the matrix material by each void falls off sufficiently sharply from the void, the voids can be regarded as isolated and non-interacting. In this case the change in macroscopic potential due to the presence of each void can be summed independently using the potential change calculated for an isolated void in an infinite matrix. Suppose, in addition, that the voids are spherical. Let $\Phi_v(\boldsymbol{\Sigma})$ denote the *change* in potential defined below due to the introduction of a spherical void in an infinite block of the matrix material subject to $\boldsymbol{\Sigma}$ at infinity. Then, assuming conditions for non-interaction are met, the macroscopic potential is given by (1.6). A similar argument can be applied to construct the potential for a material containing a dilute level of similarly shaped, aligned cracks.

3. Potential for an isolated void in an infinite matrix

3.1. Relation between potential and minimum principle for the boundary value problem

An arbitrarily shaped void occupies the region V_v in an infinite block of material which is subject to remote stress $\boldsymbol{\Sigma}$ at infinity. The relevant potential function for the void is the *change* in $\int\phi\,\mathrm{d}V$ due to introduction of the void (normalized by its volume), i.e.,

$$\Phi_v(\boldsymbol{\Sigma}) = \frac{1}{V_v}\left\{\int_{V_M}(\phi(\boldsymbol{\sigma}) - \phi(\boldsymbol{\Sigma}))\,\mathrm{d}V - V_v\phi(\boldsymbol{\Sigma})\right\}, \tag{3.1}$$

where $\boldsymbol{\sigma}$ denotes the stress field in the region V_M outside the void. From dimensional considerations, Φ_v is independent of the volume of the void. The strain-rate quantity $\partial\Phi_v/\partial\Sigma_{ij}$ can be thought of as the strain-rate contribution from an isolated void of unit volume to the overall strain-rate. In particular, $\partial\Phi_v/\partial\Sigma_m$ is the volume expansion-rate of the void.

There is a close and important connection between Φ_v and the minimum principle for the velocities for the isolated void problem as stated by Budiansky, Hutchinson and Slutsky (1982). With $\dot{\boldsymbol{E}}$ as the remote strain-rate associated with $\boldsymbol{\Sigma}$, let

$$v_i = \dot{E}_{ij}x_j + v_i^* \quad \text{and} \quad \dot{\epsilon}_{ij} = \dot{E}_{ij} + \dot{\epsilon}_{ij}^* \tag{3.2}$$

with $\dot{\epsilon}_{ij}^* = \frac{1}{2}(v_{i,j}^* + v_{j,i}^*)$ so that the additional starred velocities and strain-rates vanish at infinity. Among all additional velocity fields decaying sufficiently rapidly at infinity, the actual field minimizes the functional

$$P(\boldsymbol{v}^*) = \int_{V_M} \left(w(\dot{\boldsymbol{\epsilon}}) - w(\dot{\boldsymbol{E}}) - \Sigma_{ij}\dot{\epsilon}_{ij}^* \right) \mathrm{d}V - \int_{A_v} \Sigma_{ij} n_j v_i^* \, \mathrm{d}A. \tag{3.3}$$

Here, $\boldsymbol{n}$ is the unit normal to the void surface A_v pointing into V_M, and $w = \sigma_{ij}\dot{\epsilon}_{ij} - \phi$ is the companion potential function of the strain-rates which provides the stress according to $\sigma_{ij} = \partial w/\partial\dot{\epsilon}_{ij}$. Using manipulations similar to those described by Budiansky et al. (1982), one can show that

$$P_{\min} = V_v\{-\Phi_v + w(\dot{\boldsymbol{E}})\}, \tag{3.4}$$

where $P_{\min}$ denotes the minimum value of P evaluated using the exact solution.

3.2. *Spherical void in power-law material* (1.2) *subject to axisymmetric remote stresses*

Budiansky et al. (1982) generated approximate but accurate solutions for a spherical void in the power-law material (1.2) under general axisymmetric remote stress states. The solution process was based on the minimum principle (3.3). Although only results for the dilatation-rate of the void were reported, values of $P_{\min}$ were computed as well. Here we will make use of separate sets of results applicable at high and low stress triaxialities (i.e., high and low ratios of mean to effective remote stresses) to develop expressions for Φ_v in the corresponding ranges of triaxialities. The results for axisymmetric remote stress states are generalized to arbitrary stress states in Section 3.3.

With x_3 as the axis of symmetry, the axisymmetric remote stresses are prescribed according to

$$\sigma_{33} \to \Sigma_{33} \equiv S, \qquad \sigma_{11} \to \Sigma_{11} \equiv T, \qquad \sigma_{22} \to \Sigma_{22} \equiv T. \tag{3.5}$$

Let

$$\Sigma_m \equiv \tfrac{1}{3}\Sigma_{kk} = \tfrac{1}{3}(S + 2T) \quad \text{and} \quad \Sigma = S - T \tag{3.6}$$

so that the remote effective stress is given by

$$\Sigma_e = |\Sigma|. \tag{3.7}$$

For axisymmetric remote stress states, all dependence of Φ_v on $\boldsymbol{\Sigma}$ can be expressed using Σ and Σ_m, since they are linear combinations of S and T. Furthermore, because Φ_v is even and homogeneous of degree $n+1$ in $\boldsymbol{\Sigma}$, Φ_v can be written quite generally as

$$\Phi_v = \alpha\sigma_0(\Sigma_e/\sigma_0)^{n+1} f(x, n), \tag{3.8}$$

where f is dimensionless. Here,

$$x = \Sigma_m / \Sigma \tag{3.9}$$

is a nondimensional measure of the triaxiality of the axisymmetric remote stresses. [1] The dilatation-rate of the void is related to f by

$$\frac{\dot{V}_v}{V_v} = \frac{\partial \Phi_v}{\partial \Sigma_m} = \alpha \left(\frac{\Sigma_e}{\sigma_0} \right)^{n-1} \frac{\Sigma}{\sigma_0} \frac{\partial f}{\partial x}. \tag{3.10}$$

Then, noting that the remote strain-rate is given by

$$\dot{\epsilon}_{33} \to \dot{E} \equiv \alpha (\Sigma_e / \sigma_0)^{n-1} \Sigma / \sigma_0 \tag{3.11}$$

one can introduce a normalized dilatation-rate as

$$\frac{1}{\dot{E}} \frac{\dot{V}_v}{V_v} = \frac{\partial f}{\partial x}(x, n). \tag{3.12}$$

3.2.1. High triaxiality approximation, $|x| \gg 1$

The high triaxiality approximation for the normalized dilatation-rate derived by Budiansky et al. (1982) is, in their notation,

$$\frac{1}{\dot{E}} \frac{\dot{V}_v}{V_v} = \tfrac{3}{2} m \left\{ \frac{3|x|}{2n} - G(n, m) \right\}^n, \tag{3.13}$$

where $m = \text{sign}(x)$ and

$$-G(n, m) = (n-1)[n + g(m)]/n^2 \tag{3.14}$$

with

$$g(1) = 0.4319 \quad \text{and} \quad g(-1) = 0.4031. \tag{3.15}$$

In the limit $n \to \infty$, corresponding to a rigid-perfectly plastic material, this expression reduces to the results of Rice and Tracey (1969),

$$\frac{1}{\dot{E}} \frac{\dot{V}_v}{V_v} = \begin{cases} 0.850 \exp\{\tfrac{3}{2} x\} & \text{for } x > 0, \\ -0.826 \exp\{-\tfrac{3}{2} x\} & \text{for } x < 0, \end{cases} \tag{3.16}$$

For $n = 1$, (3.13) reduces to the exact result. When multiplied by $\dot{E}$, it also gives the exact result for $\dot{V}_v$ for purely hydrostatic loading (i.e., $\Sigma = 0$, $\Sigma_m \neq 0$) for all n. For all $n > 1$, (3.13) gives an excellent approximation for $|x| \geqslant 2$ and is even fairly good for $|x|$ as small unity, as can be seen in Fig. 5.1(b) of the paper by Budiansky et al. (1982).

By (3.12), the behavior of f for large $|x|$ consistent with (3.13) is

$$f \cong f_H(x, n) = \frac{n}{n+1} \left\{ \frac{3|x|}{2n} - G(n, m) \right\}^{n+1}, \tag{3.17}$$

where, as before, $m = \text{sign}(x)$.

[1] The distinction between x in (3.9) and X in (1.10) is deliberate. It is not true, in general, that the axisymmetric results can be represented by (1.9), although later it will be argued that they are well approximated by (1.9).

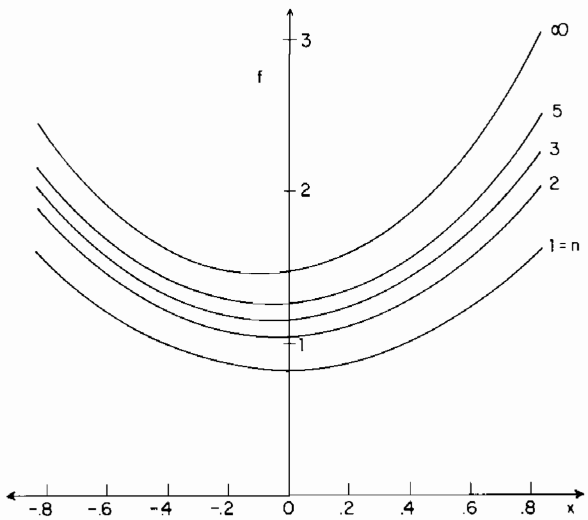

Fig. 1. $f(x, n)$ for axisymmetric remote stresses in the low triaxiality range.

3.2.2. Low triaxiality approximation, $|x| \ll 1$

The function $f(x, n)$ is plotted in Fig. 1 for x in the range $|x| \leqslant \frac{3}{4}$ for five n-values in the range from 1 to ∞. For $n = 1$,

$$f(x, 1) = \tfrac{5}{6} + \tfrac{9}{8}x^2, \tag{3.18}$$

which is exact for all x. The numerical values of f for $n > 1$ were obtained from (3.4) using the numerical results for $P_{\min}$ of Budiansky et al. (1982).

For $n > 1$, the minimum of f with respect to x occurs at $x^*(n)$ which is slightly less than zero. Denote the minimum value of f by $f^*(n)$ and let $\kappa(n)$ denote the second derivative of f with respect to x at x^*. For x in the neighborhood of x^*, we approximate f by the first two nonzero terms in its Taylor series expansion about x^*, i.e.,

$$f \equiv f_L(x, n) = f^*(n) + \tfrac{1}{2}\kappa(n)[x - x^*(n)]^2. \tag{3.19}$$

Values of f^*, κ and x^* are given in Table 1. These values were determined by fitting (3.19) to the numerical results on which Fig. 1 is based using values at $x = 0$ and $\pm\frac{1}{6}$.

By (3.12), the normalized dilatation-rate predicted from the low triaxiality approximation (3.19) is

$$\frac{1}{\dot{E}}\frac{\dot{V}_v}{V_v} = \kappa(n)[x - x^*(n)]. \tag{3.20}$$

This linear dependence is indicated by dashed lines in Fig. 2. The solid line curves in this figure are the accurate numerical results for the normalized dilatation-rate.[2] For $n > 1$, (3.20) departs significantly from the accurate results only when $|x| > \frac{1}{2}$, and for some purposes (3.20) might be sufficiently accurate for $|x|$ as large as $\frac{3}{4}$ or even 1. Since uniaxial tension and compression correspond to $x = \frac{1}{3}$, the range of applicability of the low triaxiality approximation is fairly substantial.

[2] The results for $x < 0$ were not presented by Budiansky et al. (1982). The results in Fig. 2 for the normalized dilatation-rate in the range $0 \leqslant x \leqslant \frac{3}{4}$ are slightly higher, and more accurate, than those presented in Fig. 5.1(a) of Budiansky et al. (1982). The present results were calculated using more independent amplitude factors in the minimization process than Budiansky et al. (1982) used.

Table 1

n	f^*	x^*	κ
1	0.833	0	2.25
1.5	0.965	−0.019	2.42
2	1.05	−0.031	2.55
3	1.16	−0.045	2.71
5	1.26	−0.058	2.88
10	1.35	−0.070	3.06
∞	1.46	−0.083	3.30

3.3. Approximations to f for arbitrary remote stresses

Our aim in this section is to give approximate formulas for the void potential Φ_v which apply for arbitrary remote stress states. Formulas will be given for both high and low stress triaxialities. In addition, an interpolation formula will be suggested which applies for all stresses.

With $\boldsymbol{\Sigma}$ as the remote stress and $\boldsymbol{S}$ as its deviator, introduce the remote effective stress Σ_e by (1.7), the remote mean stress invariant Σ_m by (1.8), and let $\bar{\Sigma} = (S_{ij}S_{ij}S_{jk})^{1/3}$ be the third invariant. Let

$$X = \Sigma_m/\Sigma_e \quad \text{and} \quad Y = \bar{\Sigma}/\Sigma_e. \tag{3.21}$$

Since the dependence of Φ_v on $\boldsymbol{\Sigma}$ is isotropic and homogeneous of degree $n+1$, Φ_v must have the general form

$$\Phi_v = \alpha\sigma_0(\Sigma_e/\sigma_0)^{n+1} f(X, Y, n). \tag{3.22}$$

We will neglect the dependence on Y in (3.22) and approximate Φ_v by (1.9). That f has some Y-dependence is evident from the fact that $f(x, n)$ for axisymmetric remote stress is not strictly even in x when $n > 1$. Only if $f(x, n)$ were even in x would it be possible to exactly reproduce the axisymmetric

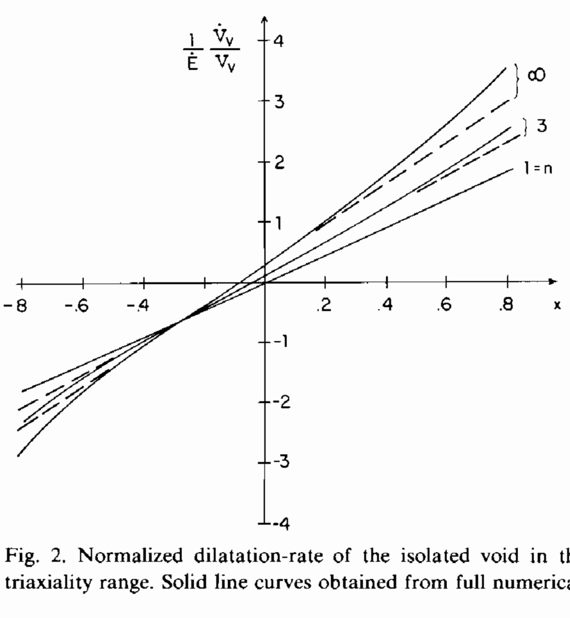

Fig. 2. Normalized dilatation-rate of the isolated void in the infinite block subject to axisymmetric remote stresses in the low triaxiality range. Solid line curves obtained from full numerical analysis. Dashed line from low triaxiality approximation (3.20).

results using a Y-independent Φ_v. [3] However, the asymmetry of the axisymmetric results with respect to x is small. For example, in (3.16) it can be seen that the two normalized dilatation-rates at corresponding values of $|x|$ differ by less than 3% in the limit $n \to \infty$, and the discrepancy is even less for finite n in the high triaxiality range. Also, the value x^* at which the minimum of f occurs, where the dilatation-rate changes sign, is only slightly displaced from zero for $n > 1$.

3.3.1. High triaxiality approximation, $|X| \gg 1$

Based on (3.17) we take

$$f(X, n) \cong f_H(X, n) = \frac{n}{n+1}\left\{\frac{3|X|}{2n} - G(n, 1)\right\}^{n+1}, \tag{3.23a}$$

where $G(n, 1)$ is given by (3.14). (Of course, $G(n, -1)$ could equally well have been used.) In the limit $n \to \infty$ this becomes

$$f \cong f_H(X, \infty) = 0.567 \exp\{\tfrac{3}{2}|X|\}. \tag{3.23b}$$

3.3.2. Low triaxiality approximation, $|X| \ll 1$

We eliminate the slight asymmetry of f with respect to x in (3.19) by setting $x^* = 0$ and take

$$f(X, n) \cong f_L(X, n) = f^*(n) + \tfrac{1}{2}\kappa(n) X^2, \tag{3.24}$$

where f^* and κ are still given by the values in Table 1. Eq. (3.24) is exact for all X when $n = 1$.

3.3.3. Interpolation approximation for all X

The following formula for f approaches f_H for large $|X|$, while for small $|X|$ it approximates f_L with a relative error of $|X|^3$:

$$f(X, n) = c_1 + c_2|X| + c_3 X^2 + \frac{n}{n+1}\left\{\frac{3|X|}{2n} - G(n, 1)\right\}^{n+1}, \tag{3.25}$$

where

$$c_1 = f^* - \frac{n}{n+1}(-G)^{n+1}, \quad c_2 = -\tfrac{3}{2}(-G)^n \quad \text{and} \quad c_3 = \tfrac{1}{2}\kappa - \tfrac{9}{8}(-G)^{n-1}, \tag{3.26}$$

For $n = 1$, the interpolation formula reduces to the exact result. The normalized dilatation-rate from (3.25) is

$$\frac{\dot{V}_v}{\dot{E}_e V_v} = \frac{\partial f}{\partial X} = 2c_3 X + \text{sign}(X)\left[\frac{3}{2}\left\{\frac{3|X|}{2n} - G\right\}^n + c_2\right], \tag{3.27}$$

where $\dot{E}_e$ is the remote effective strain-rate $(\frac{2}{3}\dot{E}_{ij}\dot{E}_{ij})^{1/2}$. The normalized dilatation-rate from (3.27) is compared with the corresponding high and low triaxiality approximations from (3.23) and (3.24) in Fig. 3 for $n = 3$.

[3] Note that $X = x$ for $S > T$ and $X = -x$ for $S < T$. Thus for axisymmetric states, the magnitudes of x and X are equal but their signs may differ.

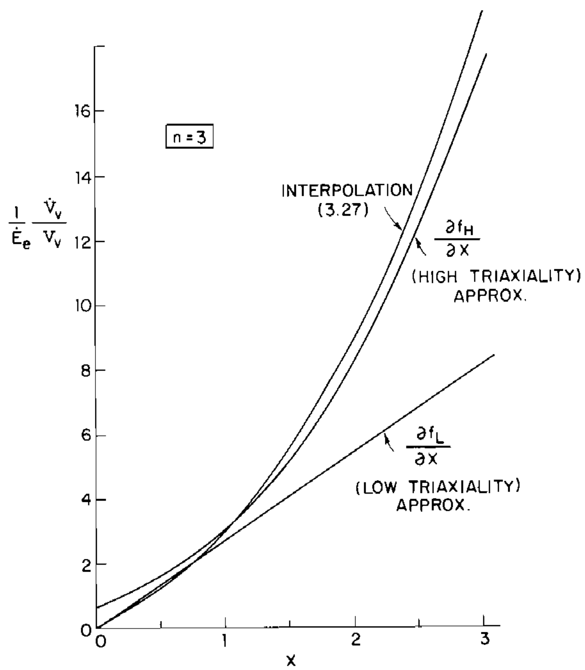

Fig. 3. Illustration of the interpolation formula (3.27).

4. Constitutive relations for various dilutely voided power-law solids

4.1. Power-law creeping solid

The potential for the power-law creeping solid in the presence of a dilute concentration of spherical voids, ρ, is given by (1.6) where our approximation for Φ_v is specified by (1.9) with f given by one of (3.23), (3.24) or (3.25) depending on the range of X. The creep-rate is given in terms of the macroscopic stress by (1.11). The creep-rate derived from the high triaxiality approximation (3.23a) is

$$\dot{E}_{ij} = \tfrac{3}{2}\alpha\left(\frac{\Sigma_e}{\sigma_0}\right)^{n-1}\frac{S_{ij}}{\sigma_0} + \rho\alpha\left(\frac{\Sigma_e}{\sigma_0}\right)^{n}\left\{\frac{3|X|}{2n} - G\right\}^{n}\left[\tfrac{3}{2}(-nG)\frac{S_{ij}}{\Sigma_e} + \tfrac{1}{2}\,\mathrm{sign}(X)\delta_{ij}\right], \tag{4.1}$$

while the result derived from the low triaxiality approximation (3.24) is

$$\dot{E}_{ij} = \tfrac{3}{2}\alpha\left(\frac{\Sigma_e}{\sigma_0}\right)^{n-1}\frac{S_{ij}}{\sigma_0} + \rho\alpha\left(\frac{\Sigma_e}{\sigma_0}\right)^{n}\left\{\tfrac{3}{2}\left[(n+1)f^* + \tfrac{1}{2}\kappa(n-1)X^2\right]\frac{S_{ij}}{\Sigma_e} + \tfrac{1}{3}\kappa X\delta_{ij}\right\}. \tag{4.2}$$

In either range of triaxiality, inspection of these formulas reveals that the contribution due to the voids to the deviatoric part of the strain-rate is much larger than the dilatational part when n is large. (By (3.14), $-nG = (n-1)[1 + (0.4319/n)]$, so that in either range the deviatoric contribution is of order n times the dilatational part when n is large.)

McMeeking (1982) has derived an expression for the power-law solid containing a dilute concentration of spherical voids under high triaxiality stress conditions. His derivation does not employ the scalar potential but instead deals directly with tensor equations. His procedure requires, in effect, a separate evaluation of the deviatoric and dilatational parts of the strain-rate. For the dilatational part he too makes use of the results of Budiansky et al. (1982). The present results are in essential agreement with McMeeking's result for that part. McMeeking's evaluation of the deviatoric part requires knowledge of the shape change of the void, which he also took from the work of Budiansky et al. (1982). This part of the calculation appears to be the most difficult to perform accurately, and at sufficiently large X his formula gives a deviatoric contribution with the incorrect sign. The great advantage of the present approach based on the scalar potential is that the deviatoric and dilatational contributions are generated together from the basic relation (1.5). Approximations are only made on the scalar potential function and not at the level involving tensor functions where approximations are generally more difficult to access.

4.2. Deformation theory solid

Suppose the matrix material is a small strain, rate-independent J_2 deformation theory solid with a power-law tensile relation

$$\epsilon = \alpha(\sigma/\sigma_0)^n \tag{4.3}$$

and a multiaxial relation

$$\epsilon_{ij} = \tfrac{3}{2}\alpha(\sigma_e/\sigma_0)^{n-1} s_{ij}/\sigma_0. \tag{4.4}$$

Because of the simple correspondence between a creeping material and a small strain, rate-independent deformation theory solid, the strain potential of the power-law deformation theory solid with a dilute concentration of spherical voids is still given Φ in (1.6). The overall strain is given by

$$E_{ij} = \partial\Phi/\partial\Sigma_{ij} \tag{4.5}$$

so that all the previous formulas apply to the voided deformation theory solid if $\dot{\boldsymbol{E}}$ is replaced by $\boldsymbol{E}$.

Approximate ways for including the effect of a strain contribution which is linear in the stress, as for a matrix material with a Ramberg–Osgood stress–strain curve, are readily generated. The simplest is to regard the power-law contribution as the 'plastic-part' and add to it the corresponding linear contribution. These possibilities will not be pursued here.

4.3. Rate-independent, flow theory solid

In this section we give the constitutive relation for a small strain, power-law flow theory (incremental) solid which coincides with the deformation theory solid (4.5) when the overall stressing is proportional, i.e., when $\boldsymbol{\Sigma} = \lambda\boldsymbol{\Sigma}^0$ where λ is increased monotonically. The solid can be regarded as rigid-plastic or, alternatively, the expression for the strain increments given below can be regarded as the plastic part of the strain increments. When the void volume concentration ρ is zero, the solid (i.e., the matrix material) is the incompressible power-law material characterized by (4.3) in simple tension, by (4.4) for proportional multiaxial stressing, and by J_2 flow theory for arbitrary stressing histories.

To construct the constitutive relation for the dilutely voided solid we take the overall yield condition in the current state as

$$\Phi(\boldsymbol{\Sigma}) = \text{const.}, \tag{4.6}$$

where Φ is again given by (1.6). If the yield condition is currently met, the condition for continued yielding is $(\partial\Phi/\partial\Sigma_{ij})\dot{\Sigma}_{ij} > 0$, where now the overdot denotes an incremental change and not necessarily the

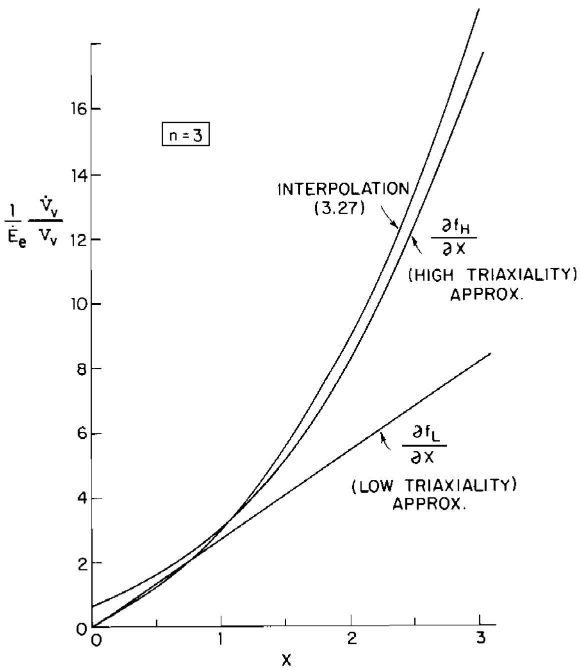

Fig. 3. Illustration of the interpolation formula (3.27).

4. Constitutive relations for various dilutely voided power-law solids

4.1. Power-law creeping solid

The potential for the power-law creeping solid in the presence of a dilute concentration of spherical voids, ρ, is given by (1.6) where our approximation for Φ_v is specified by (1.9) with f given by one of (3.23), (3.24) or (3.25) depending on the range of X. The creep-rate is given in terms of the macroscopic stress by (1.11). The creep-rate derived from the high triaxiality approximation (3.23a) is

$$\dot{E}_{ij} = \tfrac{3}{2}\alpha\left(\frac{\Sigma_e}{\sigma_0}\right)^{n-1}\frac{S_{ij}}{\sigma_0} + \rho\alpha\left(\frac{\Sigma_e}{\sigma_0}\right)^{n}\left\{\frac{3|X|}{2n} - G\right\}^{n}\left[\tfrac{3}{2}(-nG)\frac{S_{ij}}{\Sigma_e} + \tfrac{1}{2}\,\mathrm{sign}(X)\delta_{ij}\right], \tag{4.1}$$

while the result derived from the low triaxiality approximation (3.24) is

$$\dot{E}_{ij} = \tfrac{3}{2}\alpha\left(\frac{\Sigma_e}{\sigma_0}\right)^{n-1}\frac{S_{ij}}{\sigma_0} + \rho\alpha\left(\frac{\Sigma_e}{\sigma_0}\right)^{n}\left\{\tfrac{3}{2}\left[(n+1)f^* + \tfrac{1}{2}\kappa(n-1)X^2\right]\frac{S_{ij}}{\Sigma_e} + \tfrac{1}{3}\kappa X\delta_{ij}\right\}. \tag{4.2}$$

In either range of triaxiality, inspection of these formulas reveals that the contribution due to the voids to the deviatoric part of the strain-rate is much larger than the dilatational part when n is large. (By (3.14), $-nG = (n-1)[1 + (0.4319/n)]$, so that in either range the deviatoric contribution is of order n times the dilatational part when n is large.)

McMeeking (1982) has derived an expression for the power-law solid containing a dilute concentration of spherical voids under high triaxiality stress conditions. His derivation does not employ the scalar potential but instead deals directly with tensor equations. His procedure requires, in effect, a separate evaluation of the deviatoric and dilatational parts of the strain-rate. For the dilatational part he too makes use of the results of Budiansky et al. (1982). The present results are in essential agreement with McMeeking's result for that part. McMeeking's evaluation of the deviatoric part requires knowledge of the shape change of the void, which he also took from the work of Budiansky et al. (1982). This part of the calculation appears to be the most difficult to perform accurately, and at sufficiently large X his formula gives a deviatoric contribution with the incorrect sign. The great advantage of the present approach based on the scalar potential is that the deviatoric and dilatational contributions are generated together from the basic relation (1.5). Approximations are only made on the scalar potential function and not at the level involving tensor functions where approximations are generally more difficult to access.

4.2. Deformation theory solid

Suppose the matrix material is a small strain, rate-independent J_2 deformation theory solid with a power-law tensile relation

$$\epsilon = \alpha(\sigma/\sigma_0)^n \tag{4.3}$$

and a multiaxial relation

$$\epsilon_{ij} = \tfrac{3}{2}\alpha(\sigma_e/\sigma_0)^{n-1} s_{ij}/\sigma_0. \tag{4.4}$$

Because of the simple correspondence between a creeping material and a small strain, rate-independent deformation theory solid, the strain potential of the power-law deformation theory solid with a dilute concentration of spherical voids is still given Φ in (1.6). The overall strain is given by

$$E_{ij} = \partial\Phi/\partial\Sigma_{ij} \tag{4.5}$$

so that all the previous formulas apply to the voided deformation theory solid if $\dot{\boldsymbol{E}}$ is replaced by $\boldsymbol{E}$.

Approximate ways for including the effect of a strain contribution which is linear in the stress, as for a matrix material with a Ramberg–Osgood stress–strain curve, are readily generated. The simplest is to regard the power-law contribution as the 'plastic-part' and add to it the corresponding linear contribution. These possibilities will not be pursued here.

4.3. Rate-independent, flow theory solid

In this section we give the constitutive relation for a small strain, power-law flow theory (incremental) solid which coincides with the deformation theory solid (4.5) when the overall stressing is proportional, i.e., when $\boldsymbol{\Sigma} = \lambda\boldsymbol{\Sigma}^0$ where λ is increased monotonically. The solid can be regarded as rigid-plastic or, alternatively, the expression for the strain increments given below can be regarded as the plastic part of the strain increments. When the void volume concentration ρ is zero, the solid (i.e., the matrix material) is the incompressible power-law material characterized by (4.3) in simple tension, by (4.4) for proportional multiaxial stressing, and by J_2 flow theory for arbitrary stressing histories.

To construct the constitutive relation for the dilutely voided solid we take the overall yield condition in the current state as

$$\Phi(\boldsymbol{\Sigma}) = \text{const.}, \tag{4.6}$$

where Φ is again given by (1.6). If the yield condition is currently met, the condition for continued yielding is $(\partial\Phi/\partial\Sigma_{ij})\dot{\Sigma}_{ij} > 0$, where now the overdot denotes an incremental change and not necessarily the time-rate of change. It is readily verified that the following incremental relation, which satisfies normality, coincides with the deformation theory for all proportional stress histories:

$$\dot{E}_{ij} = \frac{n}{n+1} \frac{1}{\Phi} \frac{\partial \Phi}{\partial \Sigma_{ij}} \frac{\partial \Phi}{\partial \Sigma_{kl}} \dot{\Sigma}_{kl}. \tag{4.7}$$

At this point it is of interest to relate the present predictions for the power-hardening solid to the corresponding predictions from Gurson's (1977) theory specialized to the same matrix hardening law. If we focus on the increment of dilatation in the range of high stress triaxiality, then the present theory from (4.7) gives

$$\dot{E}_{kk} = \tfrac{3}{2}\rho \dot{E}_{\mathrm{e}} \operatorname{sign}(X)\left\{ \frac{3|X|}{2n} - G \right\}^{n}, \tag{4.8}$$

where f has been taken as in (3.23a). Here, it is assumed that $\dot{E}_{\mathrm{e}} \equiv \alpha n (\Sigma_{\mathrm{e}}/\sigma_0)^{n-1}(\dot{\Sigma}_{\mathrm{e}}/\sigma_0)$ is not zero, and terms of order ρ^2 have dropped in arriving at (4.8) from (4.7). The corresponding result from Gurson's theory when specialized to a dilute concentration of voids is, in the present notation,

$$\dot{E}_{kk} = \tfrac{3}{2}\rho \dot{E}_{\mathrm{e}} \sinh\left(\tfrac{3}{2}X\right). \tag{4.9}$$

Except for the lead coefficient $\frac{3}{2}$ rather than 1.70, this result is what one obtains from the suggested approximation of Rice and Tracey (1969) based on their analysis of an isolated spherical void in a perfectly plastic matrix. The strain-hardening characteristics of the matrix do not enter into this expression for the Gurson material in the dilute limit, except indirectly through the dependence of $\dot{E}_{\mathrm{e}}$ on $\dot{\Sigma}_{\mathrm{e}}$. The corresponding expression (4.8) from the present theory has a strong n-dependence. In the limit $n \to \infty$ corresponding to perfect plasticity, (4.8) becomes

$$\dot{E}_{kk} = 0.850 \rho \dot{E}_{\mathrm{e}} \operatorname{sign}(X) \exp\left\{ \tfrac{3}{2}|X| \right\}, \tag{4.10}$$

and, except for the 12% difference in the lead numerical coefficient, this agrees reasonably closely with (4.9) for $|X| \geqslant 1$. McMeeking (1982) has also discussed the dilutely voided, perfectly plastic material and has shown that his formulation is in general agreement with the dilute limit of the Gurson model insofar as the dilatation is concerned. However, the rate of growth of the voids in the strain hardening material may be substantially less than what is predicted by (4.9). For example, when $X = 2$ and $n = 5$, the dilatation-rate from (4.8) is less than one half the prediction of the Gurson theory (4.9). More extensive comparisons can be inferred from Fig. 5.1 of Budiansky et al. (1982).

Based on the above observations, it does appear that the Gurson model will substantially overestimate the dilatational component of the strain, and presumably the additional deviatoric component as well, in a dilutely voided solid with moderate strain hardening. Of course, the Gurson model was formulated to deal with void volume concentrations which may exceed dilute levels. It is possible that at non-dilute void concentrations there may be some compensation for the effect of strain hardening.

Tvergaard (1981) has attempted to improve the accuracy of the Gurson model by adjusting certain of its numerical coefficients so that the model more accurately reproduces detailed calculations for a voided material. He found that the adjusted coefficients do depend on the strain-hardening exponent, but no systematic study of this dependence was made. There appears to be room for further improvement in the Gurson model with respect to the way strain-hardening has been incorporated. Improvement along these lines is likely to be important since one of the major uses of the model is in the study of plastic flow localization. Such instability processes depend rather sensitively on strain hardening.

5. A finite shell model for non-dilute void concentrations

As a model voided material, consider a thick spherical shell with inner radius a and outer radius b made of the power-law viscous material (1.1)–(1.3). The surface of the void at $r = a$ is traction-free and uniform axisymmetric tractions are imposed over the outer surface at $r = b$. Specifically, with $\boldsymbol{\Sigma}$ denoting the stress tensor specified in terms of S and T in (3.5), the tractions imposed on the outer surface of the shell model are $\Sigma_{ij} n_j$ where $\boldsymbol{n}$ is the unit outward normal to the surface. The stress $\boldsymbol{\Sigma}$ is regarded as the macroscopic stress carried by the model material. The void volume fraction is $\rho = (a/b)^3$.

As in the analysis of the isolated void in the infinite block under axisymmetric loading, we define x as in (3.9) and we continue to define $\dot{E}$ as a reference strain-rate as in (3.11). The normalized dilatation-rate of the spherical shell model is defined as $(\dot{V}/V)/\dot{E}$ where $V = \frac{4}{3}\pi b^3$ is the current volume of the voided material element. Previously the volume of the void itself, V_v, was used to define the normalized dilatation-rate. Of course, $\dot{V} = \dot{V}_v$, since the shell material is incompressible. Thus, $(\dot{V}/V)/\dot{E} = \rho(\dot{V}_v/V_v)/\dot{E}$. For the power-law material, this normalized dilatation-rate has the functional form

$$\frac{1}{\dot{E}}\frac{\dot{V}}{V} = F(x, \rho, n). \tag{5.1}$$

In the dilute limit, as ρ becomes small, (5.1) necessarily reduces to the result obtainable from (3.12) for the isolated void in the infinite block, i.e.,

$$\frac{1}{\dot{E}}\frac{\dot{V}}{V} \to \rho\frac{\partial f}{\partial x}(x, n) \quad \text{as } \rho \to 0. \tag{5.2}$$

Numerical calculations of $F(x, \rho, n)$ have been carried out using a Rayleigh–Ritz procedure similar to that used and detailed by Budiansky et al. (1982). In the present calculations the minimum principle for finite regions was used and more Ritz functions were needed to achieve accuracy comparable to that for the infinite problem.

Plots of the normalized dilatation-rate (5.1) as a function of ρ are given in Fig. 4 for $n = 1$ and $n = 5$ for a low triaxiality level $x = \frac{1}{3}$, uniaxial tension, and a higher triaxiality level $x = 2$. The dilute approximation (5.2) with a linear ρ-dependence is also shown. The dilute approximation is fairly accurate over the entire range of ρ shown for the linear material, $n = 1$. However, the dilute approximation already significantly underestimates the normalized dilatation-rate of the finite sphere model for the nonlinear material with $n = 5$ at a void volume fraction of $\rho = 0.01$, and by $\rho = 0.02$ the discrepancy is about 50%. The interaction effect, to the extent it is modeled by the finite sphere model, is much stronger for the nonlinear material than the linear material. This is especially evident under hydrostatic loading with $S = T$. Then, from Budiansky et al. (1982),

$$\frac{\dot{V}}{V} = \tfrac{3}{2}\rho\alpha\,\mathrm{sign}(S)\left(\frac{3}{2n}\left|\frac{S}{\sigma_0}\right|\right)^n\left[1 - \rho^{1/n}\right]^{-n} = \tfrac{3}{2}\rho\alpha\,\mathrm{sign}(S)\left(\frac{3}{2n}\left|\frac{S}{\sigma_0}\right|\right)^n\left[1 + n\rho^{1/n} + \cdots\right]. \tag{5.3}$$

Under hydrostatic loading the lowest order correction to the dilute approximation from (5.3) is not linear in ρ but of order $\rho^{1/n}$. A void volume fraction of $\rho = 10^{-3}$ already increases the dilatation-rate by almost 30% above the dilute prediction when $n = 3$.

Based on analytical considerations, we expect that the lowest order correction to the dilute approximation will be linear in ρ for overall stress states which are not purely hydrostatic. Nevertheless, the finite sphere model does illustrate the fact that interaction among the voids and departures from the dilute approximation must be expected at void volume fractions as small as one percent in the strongly nonlinear materials. While this clearly places a severe limit on the applicability of the dilute approximation, it does

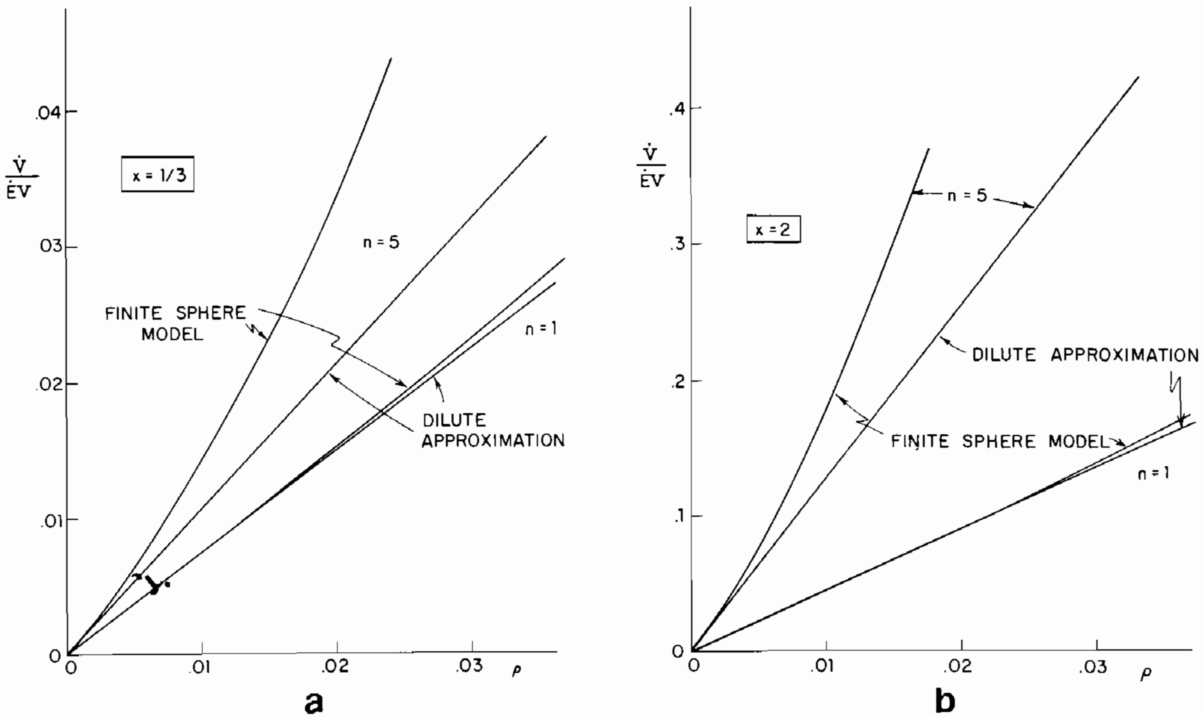

Fig. 4. Comparison of normalized dilatation-rate for finite sphere model with predictions based on dilute approximation. (a) Low triaxiality example, $x = \frac{1}{3}$. (b) High triaxiality example, $x = 2$.

not eliminate its usefulness altogether. Some of the most important flow localization phenomena precipitated by void nucleation and growth set in at low void volume fractions, often in the range of ρ between 10^{-3} and 10^{-2}.

6. Potential for power-law material with a dilute concentration of aligned penny-shaped cracks

For completeness we reproduce some related results from Hutchinson (1983) for the constitutive behavior of the power-law material (1.2) weakened by a dilute concentration of aligned penny-shaped cracks. The cracks are all taken to be perpendicular to the x_3 axis and attention is limited to overall stress states $\boldsymbol{\Sigma}$ which are axisymmetric with respect to this axis and which are again characterized by S and T in (3.5).

An approximate formula for the potential change resulting from the introduction of an isolated penny-shaped crack of radius a (aligned perpendicular to the x_3 axis) in an infinite block of material subject to remote axisymmetric stress $\boldsymbol{\Sigma}$ is

$$\Phi_c(\boldsymbol{\Sigma}) = 4\alpha\sigma_0 a^3(1 + 3/n)^{-1/2}(\Sigma_e/\sigma_0)^{n+1}(S/\Sigma_e)^2. \tag{6.1}$$

Here, Φ_c is conveniently defined as $\int_{V_M}(\phi(\boldsymbol{\sigma}) - \phi(\boldsymbol{\Sigma}))\,dV$. This formula is exact for $n = 1$, and, as discussed by Hutchinson (1983), is highly accurate for all n for $|S/\Sigma_e| \leqslant 2$. It becomes inaccurate in the high triaxiality range for S/Σ_e larger than about 3 or 4.

The potential for the dilutely cracked material is

$$\Phi = \frac{\alpha\sigma_0}{n+1}\left(\frac{\Sigma_e}{\sigma_0}\right)^{n+1}\left\{1+\rho\left(\frac{S}{\Sigma_e}\right)^2\right\}, \tag{6.2}$$

where, from (6.1), the crack density parameter is

$$\rho = 4a^3N(n+1)(1+3/n)^{-1/2}, \tag{6.3}$$

where N is the number of cracks of radius a per unit volume. The strain-rate from (1.5) is

$$\dot{E}_{ij} = \alpha\left(\frac{\Sigma_e}{\sigma_0}\right)^n\left\{\frac{3}{2}\frac{S_{ij}}{\Sigma_e}+\rho\left[\frac{3(n-1)}{2(n+1)}\frac{S_{ij}}{\Sigma_e}\left(\frac{S}{\Sigma_e}\right)^2+\frac{2}{n+1}\frac{S}{\Sigma_e}m_{ij}\right]\right\}, \tag{6.4}$$

where $m_{ij} = \delta_{i3}\delta_{j3}$.

The expression for the strain-rate (6.4) is limited to axisymmetric stress states since (6.1) is derived under that restriction. While (6.4) is anisotropic as it stands, its application to creep-constrained grain boundary cavitation of polycrystalline materials (Hutchinson, 1983) assumed that the penny-shaped microcracks always appear perpendicular to the maximum principle stress S. With this interpretation the material characterized by (6.4) is isotropic.

Acknowledgment

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