WRINKLING OF CURVED THIN SHEET METAL

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ABSTRACT

Conditions for the onset of wrinkling in doubly-curved sheet metal undergoing forming are obtained from a plastic buckling analysis for short-wavelength, shallow modes. The region of the sheet susceptible to wrinkling is assumed to be unconstrained by the die. When the principal axes of the membrane stress state coincide with the principal axes of the curvatures, simple formulas for the stresses or strains at wrinkling are obtained.

1. INTRODUCTION

Wrinkling is increasingly becoming one of the most common and troublesome modes of unacceptable deformation in sheet metal forming. Wrinkling can be viewed as a plastic buckling process in which the wavelength of the mode in one direction is extremely short. The mode is a local one which depends on the local curvatures and thickness of the sheet, on its material properties, and on the stress state. In this paper we carry out a plastic buckling analysis of local wrinkling by exploiting the fact that the short wavelength modes are shallow and can be analyzed using shallow shell theory, or equivalently, Donnell-Mushtari-Vlasov (DMV) theory. The wrinkling phenomenon is closely related to certain shell buckling modes, and existing knowledge on the plastic buckling of shells is helpful in understanding and predicting wrinkling.

2. WRINKLING ANALYSIS OF DOUBLY-CURVED THIN SHEETS

Our approach consists of formulating the problem within the context of plastic bifurcation theory for thin plates and shells following a treatment of plastic buckling using Donnell-Mushtari-Vlasov (DMV) shallow shell theory. This allows us to determine the critical conditions for buckling in a short wavelength or "wrinkling" mode. The analysis is restricted to

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modes for which the characteristic wavelength is large compared to the sheet thickness, yet small compared to the radii of curvatures of the sheet. Such modes are termed "shallow" and are accurately described by the DMV theory.

We imagine in the current stage of the forming process that the sheet has attained a doubly-curved state with principal radii of curvatures R_1 and R_2 which are assumed to be constant over the region of the sheet being examined for susceptibility to the shallow local modes. We limit our investigation to regions of the sheet which are not in contact with the die and thus neglect any interaction between sheet and die. Several further idealizations are made which help to simplify the analysis.

- At the beginning of the current stage of the forming process the sheet material is assumed to be isotropic in the unstressed state and uniform over the region being examined with no variation through the thickness. The material will be assumed to be characterized by either J_2 deformation theory or J_2 flow theory when stressed into the plastic range.

- During the current forming process the prebuckling, or prewrinkling, state of stress in the sheet is assumed to be a membrane state which is uniform over the region being examined for wrinkling.

- The principal axes of the prebuckling membrane state are assumed to coincide with the principal axes of the curvature of the sheet. Let x_1 and x_2 denote surface coordinates everywhere aligned with these principal axes and let σ_1 and σ_2 denote the principal membrane stresses.

A fairly complete discussion of plastic buckling and the basic relations for the DMV theory of plates and shells can be found in [1]. According to this theory and the shallow shell approximations, buckling from the uniform membrane state gives rise to the following incremental stretching $(\dot{E}_{\alpha\beta})$ and bending $(\dot{K}_{\alpha\beta})$ strains

$$\begin{aligned}\dot{E}_{\alpha\beta} &= \tfrac{1}{2}(\dot{U}_{\alpha,\beta} \quad \dot{U}_{\beta,\alpha}) + b_{\alpha\beta}\dot{W} \\ \dot{K}_{\alpha\beta} &= -\dot{W}_{,\alpha\beta}\end{aligned} \tag{2.1}$$

Here $\dot{U}_\alpha$ $(\alpha,\beta = 1,2)$ are the incremental displacements in the x_1, x_2 directions, $\dot{W}$ is the incremental buckling displacement normal to the middle surface of the sheet, $b_{\alpha\beta}$ is the curvature tensor of the middle surface in the prebuckling state and a comma denotes covariant differentiation with respect to a surface coordinate. The above incremental strains lead to incremental stress resultants $(\dot{N}_{\alpha\beta})$ and bending moments $(\dot{M}_{\alpha\beta})$ at buckling. These are given by

$$\dot{N}_{\alpha\beta} = t\bar{L}^{\alpha\beta\kappa\gamma}\dot{E}_{\kappa\gamma}$$

$$\dot{M}_{\alpha\beta} = \frac{t^3}{12}\bar{L}^{\alpha\beta\kappa\gamma}\dot{K}_{\kappa\gamma} \tag{2.2}$$

where t is the current sheet thickness and $\bar{\underset{\sim}{L}}$ are the plane-stress incremental moduli relating stress increments $\dot{\tau}^{\alpha\beta}$ to strain increments $\dot{\eta}_{\alpha\beta}$ through $\dot{\tau}^{\alpha\beta} = \bar{L}^{\alpha\beta\kappa\gamma}\dot{\eta}_{\kappa\gamma}$. These moduli, to be specified later, are uniform throughout the sheet because of our assumption of a homogeneous membrane state prior to buckling.

To determine the critical stress state for buckling we consider the following "bifurcation functional" [1]

$$F(\dot{\underset{\sim}{U}},\dot{W}) = \int_S \left[\frac{t^3}{12}\bar{L}^{\alpha\beta\kappa\gamma}\dot{K}_{\alpha\beta}\dot{K}_{\kappa\gamma} + t\bar{L}^{\alpha\beta\kappa\gamma}\dot{E}_{\alpha\beta}\dot{E}_{\kappa\gamma} + N^{\alpha\beta}\dot{W}_{,\alpha}\dot{W}_{,\beta}\right]dS \tag{2.3}$$

where S is the region of the sheet middle surface over which the wrinkles occur. The condition that $F > 0$ for all admissible fields $\dot{U}_\alpha, \dot{W}$ ensures that bifurcation will not occur. Conversely, bifurcation first becomes possible when $F = 0$ for some non-zero field. For wrinkling or bifurcation in short-wavelength, shallow modes we consider the following fields:

$$\dot{W} = At\cos(\lambda_1 x_1/\ell)\cos(\lambda_2 x_2/\ell)$$

$$\dot{U}_1 = Bt\sin(\lambda_1 x_1/\ell)\cos(\lambda_2 x_2/\ell) \tag{2.4}$$

$$\dot{U}_2 = Ct\cos(\lambda_1 x_1/\ell)\sin(\lambda_2 x_2/\ell)$$

where

$$\ell = \sqrt{Rt} \tag{2.5}$$

and R will later be identified with either R_1 or R_2, as appropriate. In (2.4) A, B and C are constants representing the relative displacement amplitudes of the mode shape and λ_1, λ_2 are nondimensional wave numbers. In employing these fields we anticipate that wrinkling occurs over a certain region S of the sheet which spans many wavelengths of the buckling mode. The boundary conditions or continuity conditions along the edges of S then become relatively unimportant. In this sense, our analysis is a local one.

The analysis involves substituting the fields (2.4) into the bifurcation functional (2.3) and integrating over S. In so doing we use the relations (2.1) with $b_{11} = 1/R_1$, $b_{22} = 1/R_2$, as well as $N^{11} = -t\sigma_1$, $N^{22} = -t\sigma_2$ and the following formulas

$$\left.\begin{aligned}&\int_S [\sin(\lambda_1 x_1/\ell)\sin(\lambda_2 x_2/\ell)]^2 dS\\&\int_S [\cos(\lambda_1 x_1/\ell)\cos(\lambda_2 x_2/\ell)]^2 dS\end{aligned}\right\} = \beta S \tag{2.6}$$

Here $\beta = 1/4$ if both λ_1 and λ_2 are nonzero and $\beta = 1/2$ if either λ_1 or λ_2 are zero. The functional (2.3) can then be written as

$$F = \beta S t \left(\frac{t}{\ell}\right)^2 \{u\}^T [M] \{u\} \tag{2.7}$$

where $\{u\} = (A,B,C)$ is the displacement-amplitude vector and the matrix $[M]$ is given by

$$\begin{aligned}
M_{11} &= \frac{1}{12}\left(\frac{t}{\ell}\right)^2 \left\{L_{11}\lambda_1^4 + L_{22}\lambda_2^4 + 2(L_{12} + 2L_{44})\lambda_1^2\lambda_2^2\right\} + \left\{L_{11}\left(\frac{\ell}{R_1}\right)^2 + L_{22}\left(\frac{\ell}{R_2}\right)^2\right.\\
&\qquad \left. + 2L_{12}\left(\frac{\ell}{R_1}\right)\left(\frac{\ell}{R_2}\right)\right\} - \left\{\sigma_1\lambda_1^2 + \sigma_2\lambda_2^2\right\}\\
M_{22} &= L_{11}\lambda_1^2 + L_{44}\lambda_2^2 \quad , \quad M_{33} = L_{22}\lambda_2^2 + L_{44}\lambda_1^2\\
M_{12} &= M_{21} = L_{11}\lambda_1\left(\frac{\ell}{R_1}\right) + L_{12}\lambda_1\left(\frac{\ell}{R_2}\right)\\
M_{13} &= M_{31} = L_{22}\lambda_2\left(\frac{\ell}{R_2}\right) + L_{12}\lambda_2\left(\frac{\ell}{R_1}\right)\\
M_{23} &= M_{32} = (L_{12} + L_{44})\lambda_1\lambda_2
\end{aligned} \tag{2.8}$$

Here we have introduced the abbreviated notation $L_{11} = \bar{L}_{1111}$, $L_{22} = \bar{L}_{2222}$, $L_{12} = \bar{L}_{1122}$, $L_{44} = \bar{L}_{1212}$. Boundary, or continuity, conditions around the perimeter of S are confined to a narrow strip of width on the order of ℓ and their effect on F, compared to (2.7), is small for small ℓ^2/S.

Buckling in the mode (2.4) is possible when the associated bifurcation functional $F = 0$. In view of (2.7) this first occurs when the determinant of $[M]$ vanishes. To determine the critical stress values $\sigma_1^{cr}, \sigma_2^{cr}$ for which short-wavelength buckling first occurs, we minimize this determinant with respect to the waveform parameters λ_1 and λ_2 and set the minimum equal to zero. The values of λ_1 and λ_2 so obtained describe the corresponding critical buckling pattern.

3. CASE OF WRINKLING ALONG A PRINCIPAL AXIS

For the prebuckling geometry and stress state considered in the previous section, wrinkling will in most cases be aligned with one of the principal curvature (stress) directions. In such cases either λ_1 or λ_2 is zero and the previous analysis simplifies considerably. For example, for wrinkles perpendicular to the x_1-direction we put $\lambda_2 = 0$ in (2.8). The bifurcation condition $\det[M] = 0$ becomes

$$D = M_{11}M_{22} - M_{12}^2 = 0 \tag{3.1}$$

The only stress component that appears explicitly in (3.1) is σ_1. Thus (3.1) gives a relation between σ_1 and λ_1 for buckling. Minimizing σ_1 with respect to λ_1 gives the following expression for the critical wrinkling stress

$$\sigma_1^{cr} = \frac{1}{\sqrt{3}}\left(\frac{t}{R_2}\right)(L_{11}L_{22} - L_{12}^2)^{1/2} \tag{3.2}$$

The corresponding critical wavelength parameter is

$$\lambda_1^{cr} = [2\sqrt{3}(L_{11}L_{22} - L_{12}^2)^{1/2}/L_{11}]^{1/2} \tag{3.3}$$

when R in (2.5) is taken as R_2 so that $\ell = \sqrt{R_2 t}$. For wrinkling perpendicular to the x_2-direction the analogous relations are obtained by simply interchanging the indices $1 \leftrightarrow 2$ in (3.2) and (3.3). Although σ_2 does not enter explicitly in the critical condition (3.2), it nevertheless does influence wrinkling since the incremental moduli $\underset{\sim}{L}$ depend on σ_2.

The result (3.2) suggests that the ratio t/R_2 is the only relevant geometric parameter for wrinkles aligned with the x_2-direction. The radius of curvature R_1 perpendicular to the wrinkles does not affect σ_1^{cr}. Similar observations obviously hold for the value σ_2^{cr} associated with wrinkles lying perpendicular to the x_2-direction.

Explicit results based on the commonly used J_2 flow and deformation theories of plasticity will now be given. As discussed in [1] each of these constitutive laws has a long history in plastic bifurcation calculations. It is generally found that the deformation theory, which predicts lower buckling stresses and strains than the corresponding flow theory, gives results which are in better agreement with experiment. For this reason, our emphasis will be on the deformation theory predictions.

To simplify the expressions, incompressibility will be assumed. For either of the J_2 theories the incremental moduli can then be expressed as follows:

$$L_{11} = \frac{4}{3}\bar{E} - (\bar{E} - E_t)\left(\frac{\sigma_1}{\sigma_e}\right)^2$$

$$L_{22} = \frac{4}{3}\bar{E} - (\bar{E} - E_t)\left(\frac{\sigma_2}{\sigma_e}\right)^2 \qquad (3.4)$$

$$L_{12} = \frac{2}{3}\bar{E} - (\bar{E} - E_t)\left(\frac{\sigma_1\sigma_2}{\sigma_e^2}\right)$$

where $\sigma_e = \sqrt{3J_2} = (\sigma_1^2 - \sigma_1\sigma_2 + \sigma_2^2)^{1/2}$ is the effective stress and E_t is the tangent modulus, i.e., the slope of the uniaxial stress-strain curve at the stress level σ_e. With flow theory $\bar{E}$ is equal to Young's modulus E while for deformation theory $\bar{E}$ corresponds to the secant modulus E_s, obtainable from the uniaxial stress-strain curve at the stress value σ_e (i.e., $E_s \equiv \sigma_e/\varepsilon(\sigma_e)$).

Substituting (3.4) in (3.2) gives

$$\sigma_1^{cr} = \frac{2}{3}\frac{t}{R_2}\sqrt{\bar{E}E_t} \quad , \quad \sigma_2^{cr} = \frac{2}{3}\frac{t}{R_1}\sqrt{\bar{E}E_t} \qquad (3.5)$$

for wrinkles aligned perpendicular to the x_1- and x_2-directions, respectively.

Consider the case of proportional loading or straining in the pre-buckling state. Let $\sigma_2/\sigma_1 = \alpha = \text{const}$ and $\varepsilon_2/\varepsilon_1 = \rho = \text{const}$ be the imposed stress and strain ratios, respectively. Both flow theory and deformation theory give

$$\frac{\sigma_1}{\sigma_e} = \frac{2+\rho}{[3(1+\rho+\rho^2)]^{1/2}} \quad , \quad \frac{\sigma_2}{\sigma_e} = \frac{1+2\rho}{[3(1+\rho+\rho^2)]^{1/2}} \qquad (3.6)$$

and $\alpha = (1+2\rho)/(2+\rho)$. The effective strain, defined as $\varepsilon_e = (2\varepsilon_i\varepsilon_i/3)^{1/2}$, is given by

$$\varepsilon_e = \frac{2(1+\rho+\rho^2)^{1/2}}{\sqrt{3}}\varepsilon_1 \qquad (3.7)$$

while $\sigma_e = (1-\alpha+\alpha^2)^{1/2}\sigma_1$. Furthermore, for a power-law hardening of the type $\sigma_e = K\varepsilon_e^N$ in the plastic range we have

$$E_t = NK\varepsilon_e^{N-1} \quad , \quad E_s = K\varepsilon_e^{N-1} \qquad (3.8)$$

These relations can be substituted in (3.5) to get explicit expressions for σ_1^{cr} or σ_2^{cr} at wrinkling. With deformation theory $(\bar{E} \equiv E_s)$ we obtain the simple results

$$\varepsilon_1^{cr} = \frac{\sqrt{N}}{(2+\rho)}\frac{t}{R_2} \quad , \quad \varepsilon_2^{cr} = \frac{\sqrt{N}}{(2+\rho^{-1})}\frac{t}{R_1} \tag{3.9}$$

for wrinkles aligned perpendicular to the x_1- and x_2-directions, respectively. Analogous expressions can be developed for J_2 flow theory, but these do not turn out to be quite as simple and attractive as the above deformation theory result.

4. DISCUSSION

Although some of the idealizations made in leading up to the formulas for the critical strains in (3.9) are rather restrictive, we nevertheless believe that these formulas are indicative of the interplay between geometry and material properties in determining wrinkling. The critical strain at wrinkling decreases with decreasing strain hardening and with decreasing ratio of thickness to radius of curvature. It is therefore not surprising that wrinkling has become a more prominent problem, since the higher strength sheet metals being favored in recent years tend to be thinner with lower strain hardening.

The wrinkling condition (3.9) depends at least as strongly on geometry, through t/R , as on strain hardening. The critical stress (3.5)

$$\sigma_1^{cr} = \frac{2}{3}\frac{t}{R_2}\sqrt{\bar{E}E_t}$$

for wrinkles with ridges perpendicular to the x_1-direction is precisely the critical compressive axial stress for a cylindrical shell of radius R_2 which undergoes axisymmetric buckling, assuming the shell material is incompressible and that a prebuckling hoop stress $\sigma_2 = \alpha\sigma_1$ is also present. Results of this type for the plastic buckling of shells were originally due to Bijlaard [2] in the late 1940's. Axisymmetric buckles occurring on axially compressed cylindrical shells are short-wavelength modes which are not unlike sheet metal wrinkles [3]. Although, because they extend around the complete circumference of the cylinder, they tend to localize in a single buckle.

The strong sensitivity of plastic buckling predictions for plates and shells to the choice of plastic constitutive law (i.e. deformation theory vs. flow theory) was recognized in the late 1940's and early 1950's as well. A full discussion of the issues is given in [1], along with basic references on the subject. It seems reasonable to suppose that J_2 deformation theory predictions for wrinkling strains will be much more realistic than those based on J_2 flow theory, especially for more-or-less proportional prewrinkling strain histories. Nevertheless, the inherent lack of any strain history dependence associated with the deformation theory points to the obvious limitations of this constitutive law. We also emphasize that another important limitation of the present analysis is the assumption that the unloaded sheet is isotropic at the start of the stage of the forming process in which wrinkling occurs.

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