ON NEUTRAL HOLES IN TAILORED, LAYERED SHEETS

B. Budiansky, J. W. Hutchinson and A. G. Evans

Division of Applied Sciences
HARVARD UNIVERSITY
Cambridge, Massachusetts 02138

August 1992
On Neural Holes in Tailored, Layered Sheets

by

B. Budiansky*, J. W. Hutchinson*, and A. G. Evans**

*Division of Applied Sciences, Harvard University, Cambridge, MA 02138
**Materials Department, University of California, Santa Barbara, CA 93106

It has been suggested that multilayered sheets, in which alternating layers have different elastic moduli, might lend themselves to tailoring to reduce, or even eliminate, harmful stress concentrations at holes or other stress raisers. Such tailoring could be implemented by making the sheet thickness spatially nonuniform, varying the number of layers, but keeping the layering pattern unchanged; or, keeping the total thickness unchanged, by varying the pattern of layer locations and thicknesses; or by a combination of these two approaches. We will call the first method "thickness tailoring", and the second "modulus tailoring". Tailored fabrication of such nonuniform layered sheets seems particularly well suited to masked deposition techniques.

This note provides a preliminary analytical assessment of the theoretical feasibility of designing a tailored, layered sheet that would alleviate the stress concentration induced by a circular hole in a field of balanced biaxial tension (see Fig. 1). If the stress concentration is actually eliminated, the result is a so-called "neutral" hole. It should be emphasized at the outset that reducing the average circumferential stress at the boundary of the hole is definitely not necessarily the desired goal. As we shall see, if modulus tailoring with constant overall thickness is exploited, and only the relative volumes of the layer constituents are changed, the stresses within the individual layers can be reduced while the average stress goes up! (This seemingly paradoxical result takes a little getting used to; the reason it's right is that while the stress in the stiffer material drops, there is more of it, so the average rises.) Conversely, a misguided reduction of the average hole-boundary stress by means of modulus tailoring can lead to higher stress concentrations within the layers.

We consider a two-constituent layered sheet, with Young's moduli $E_\alpha$ ($\alpha=1,2$) in the alternating layers, and for simplicity, we assume the same Poisson's ratio $\nu$ in each layer. The effective sheet modulus is $E=f_1E_1 + f_2E_2$, where the $f_i$s are volume fractions. At each $r$, denote the average radial and circumferential stresses by $\sigma_r$ and $\sigma_\theta$, and let $\sigma_r^{(\alpha)}$, $\sigma_\theta^{(\alpha)}$ ($\alpha=1,2$) be the stresses in the layers. The stress-strain relations are

\[
\varepsilon_r = \frac{\sigma_r^{(\alpha)} - \nu \sigma_\theta^{(\alpha)}}{E_\alpha} = \frac{\sigma_r - \nu \sigma_\theta}{E(r)}
\]

\[
\varepsilon_\theta = \frac{\sigma_\theta^{(\alpha)} - \nu \sigma_r^{(\alpha)}}{E_\alpha} = \frac{\sigma_\theta - \nu \sigma_r}{E(r)}
\]
Let

\[ \sigma_r = \frac{E(r)}{E(\infty)} s_r, \quad \sigma_\theta = \frac{E(r)}{E(\infty)} s_\theta \]  

(2)

where \( E(\infty) \) is the untailored sheet modulus far from the hole. Then

\[ \sigma_r^{(a)} = \frac{E_a}{E(\infty)} s_r, \quad \sigma_\theta^{(a)} = \frac{E_a}{E(\infty)} S \]  

(3)

and so the layer stresses are proportional to \( s_r \) and \( s_\theta \). Hence, it is the value of \( s_\theta \) at \( r=\alpha \) that we must seek to lower by tailoring \( E(\alpha) \), or the sheet thickness \( h(\alpha) \), or both. Note that while the stress concentration factor (SCF) for the average sheet stress \( \sigma_\theta \) is \( \sigma_\theta(\alpha)/S \), the layer concentration factors are

\[ \frac{\sigma_\theta^{(a)}(\alpha)}{\sigma_\theta^{(a)}(\infty)} = \frac{s_\theta(\alpha)}{s_\theta(\infty)} = \frac{s_\theta(\alpha)}{S} \]  

(4)

For a uniform layered sheet, these layer concentration factors are equal to the classical stress concentration factor \( \sigma_\theta(\alpha)/S = 2 \).

The equations of equilibrium and compatibility are

\[ \frac{d(rh\sigma_r)}{dr} = h\sigma_\theta \]  

(5)

and

\[ \frac{d(r\varepsilon_\theta)}{dr} = \varepsilon_r \]  

(6)

respectively. These may be rewritten as

\[ \left[ \lambda \rho s_r \right]' = \lambda s_\theta \]  

(7)

\[ \left[ \rho(s_\theta - vs_r) \right]' = s_r - vs_\theta \]  

(8)

in terms of \( \rho = r/a \), and the tailoring function defined by
\[
\lambda(\tau) = \frac{E(\tau) h(\tau)}{E(\infty) h(\infty)}
\]

(9)
Primes denote derivatives with respect to \( \rho \).

We proceed in a semi-inverse fashion by asserting the spatial distribution
\[
s_\tau = S(1 - \rho^{n-1})
\]
(10)
and solving the compatibility equation (8) for \( s_0 \) to get
\[
s_0 = S \left[ 1 + \frac{1 - v(n-1)}{(n-1-v)\rho^n} - \frac{C}{\rho^{(1+v)}} \right]
\]
(11)
where \( C \) is a constant. The only value of \( C \) that leads to a bounded tailoring function is
\[
C = \frac{2n - n^2}{n - 1 - v}
\]
(12)
and this gives the layer stress concentration factor \( s_0/S = n \) at \( \rho = 1 \). The tailoring formula
\[
\lambda(\tau) = \exp \left[ \frac{n(2-n)}{n-1-v} \int_0^{1/\tau} \frac{x^v - x^{n-1}}{1-x^n} \, dx \right]
\]
(13)
follows from the equilibrium equation (7). In all cases the peak value of \( \lambda(\tau) \), as expected, occurs at \( \tau = a \), and is given by
\[
\lambda(a) = \exp \left[ \frac{n(2-n)}{n-1-v} \int_0^1 x^v - x^{n-1} \, dx \right] \quad (n \neq 1 + v)
\]
\[
= \exp \left[ -(1-v^2) \int_0^1 \frac{x^v \log x}{1-x^n} \, dx \right] \quad (n = 1 + v)
\]
(14)
For \( v = 0 \) this last result equals \( \exp(\pi^2/6) \).

Fig. 2 shows how the peak tailoring magnitude varies with the layer stress concentration factor \( n \), for several values of \( v \). We remark that if only thickness tailoring is used, the SCF for average stress is the same as that for the layers, and so is also reduced below 2. But for pure modulus tailoring, the SCF for the average stress is given by \( n\lambda(a) \), and this always exceeds 2 for \( n < 2 \).

To get a neutral hole, we set \( n = 1 \) in the formula for \( \lambda(\tau) \), and find
\[
\lambda_{\text{neutral}}(\tau) = \exp \left[ \frac{1}{v} \int_0^{1/\tau} \frac{1-x^v}{1-x} \, dx \right]
\]
(15)
For \( v = 0 \), this result becomes
\[
\lambda_{\text{neutral}}(\tau) = \exp \left[ -\int_0^{1/\tau} \frac{x^v \log x}{1-x} \, dx \right] \quad (v = 0)
\]
(16)
Fig. 3 shows how \( \lambda_{\text{neutral}} \) varies with \( \tau/a \) for \( v = 0, 1/4, \) and \( 1/2 \).

We should check the values of \( \sigma_0^{(\alpha)}(\tau)/\sigma_0^{(\alpha)}(\infty) = s_0(\tau)/S \) away from the hole. In the case of the neutral hole, we find

-3-
\[ \frac{s_0}{S} = 1 - \left( \rho^{-1} - \rho^{-(1+v)} \right)/v \quad (v \neq 0) \]
\[ = 1 - \rho^{-1} \log \rho \quad (v = 0) \]

and so the peak layer stress does indeed occur at the hole.

Acknowledgements

This work was supported by DARPA's Defense Sciences Research Council, under contract to the University of Michigan, by a DARPA URI grant to the University of California at Santa Barbara, and by the Division of Applied Sciences, Harvard University.
Fig. 2. Tailoring function at hole boundary vs. layer stress concentration factor.

Fig. 3. Spatial variation of tailoring function for a neutral hole.