Journal of Applied Mechanics

Brief Notes

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Asymmetric Four-Point Crack Specimen

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Accurate results for the stress intensity factors for the asymmetric four-point bend specimen with an edge crack are presented. A basic solution for an infinitely long specimen loaded by a constant shear force and a linear moment distribution provides the reference on which the finite geometry solution is based.
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This note was prompted by a comparison ([1]) of existing numerical solutions ([2–4]) for the crack specimen known as the asymmetric four-point specimen shown in Fig. 1. Discrepancies among the solutions are as large as 25 percent within the parameter range of interest. Moreover, in some instances the full set of nondimensional parameters specifying the geometry (there are four) have not been reported. The specimen has distinct advantages for mixed mode testing, including the determination of mixed mode fatigue crack thresholds. Here a new fundamental reference solution is given for a infinitely long cracked specimen subject to a constant shear force and associated bending moment distribution. The small corrections needed to apply this solution to the finite four-point loading geometry are included.

By static equilibrium (the configuration in Fig. 1 is statically determinant), the shear force, Q, between the inner loading points and the bending moment, M, at the crack are related to the force, P, by (all three quantities are defined *per unit thickness*):

$$Q=P(b_2-b_1)/(b_2+b_1) \quad \text{and} \quad M=cQ. \tag{1}$$

Consider first the reference problem of an infinite specimen with crack of length a subject to a constant shear force Q and associated linearly varying bending moment M. In the absence of the crack, the exact solution for the cross section has a parabolic distribution of shear stress proportional to Q and a linear variation of normal stress proportional to M ([5]). By superposition of these two contributions, the solution for the intensity factors in the presence of the crack can be written exactly in the form

$$K_{\mathrm{I}}^R=\frac{6cQ}{W^2}\sqrt{\pi a}F_{\mathrm{I}}(a/W) \tag{2a}$$

$$K_{\mathrm{II}}^R=\frac{Q}{W^{1/2}}\frac{(a/W)^{3/2}}{(1-a/W)^{1/2}}F_{\mathrm{II}}(a/W) \tag{2b}$$

where, anticipating the application, we have taken $M=cQ$ at the crack. The solution (2a) is the same as that for a pure moment. It has been obtained numerically to considerable accuracy. Tada et al. [6] give

$$F_{\mathrm{I}}\left(\frac{a}{W}\right)=\sqrt{\frac{2W}{\pi a}\tan\frac{\pi a}{2W}}\;\frac{0.923+0.199\left(1-\sin\frac{\pi a}{2W}\right)^4}{\cos\frac{\pi a}{2W}}$$

$$\text{for } 0\leq\frac{a}{W}\leq 1 \tag{3a}$$

while Murakami [7] gives

$$F_{\mathrm{I}}\left(\frac{a}{W}\right)=1.122-1.121\left(\frac{a}{W}\right)+3.740\left(\frac{a}{W}\right)^2+3.873\left(\frac{a}{W}\right)^3$$
$$-19.05\left(\frac{a}{W}\right)^4+22.55\left(\frac{a}{W}\right)^5 \quad \text{for } \frac{a}{W}\leq 0.7. \tag{3b}$$

The second solution (2b) is not in the literature.

Finite element analyses of the reference problem have been carried out to obtain both F_{I} (as a check) and F_{II}. Our results for F_{I} agree with (3b) to four significant figures over the entire range

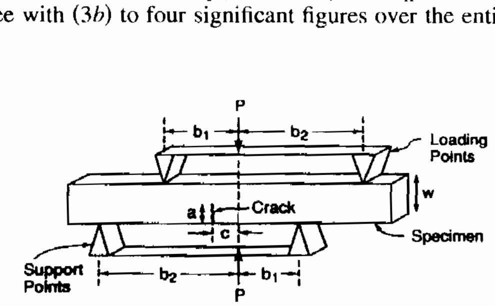

Fig. 1 Geometry of the asymmetric bending and shear specimen

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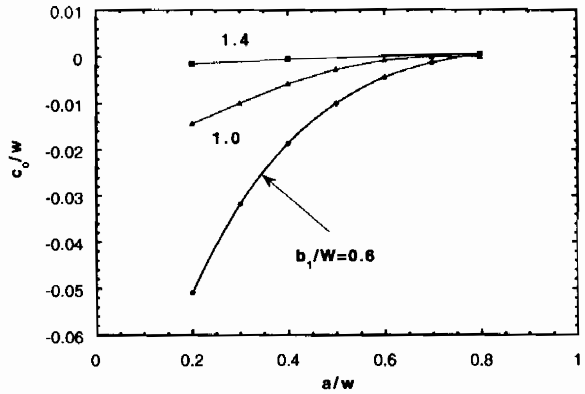

Fig. 2 Location of the crack for pure mode II at its tip ($\alpha=1$)

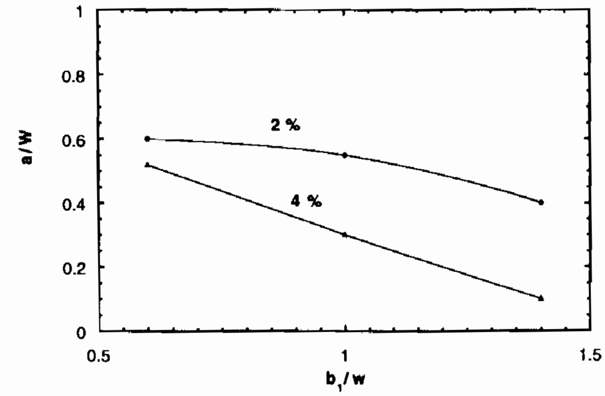

Fig. 4 Error boundaries for mode II stress intensity factor of two percent and four percent for ($\alpha=1$) for the reference solution (2). Combinations ($a/W, b_1/W$) lying above a boundary have smaller error.

of a/W indicated. Equation ($3a$) appears to be less accurate over this same range (with error less than two percent), but it can be used for $a/W>0.7$. The same finite element meshes were used to compute F_{II}. The following polynomial representation was obtained by fitting the numerical results:

$$F_{\mathrm{II}}\left(\frac{a}{W}\right)=7.264-9.37\left(\frac{a}{W}\right)+2.74\left(\frac{a}{W}\right)^2+1.87\left(\frac{a}{W}\right)^3$$

$$-1.04\left(\frac{a}{W}\right)^4 \quad \text{for } 0\leqslant\frac{a}{W}\leqslant 1. \tag{4}$$

This result is believed to be accurate to within one percent over the entire range of a/W. The results of Suresh et al. [4] determined for a specific choice of the other dimensional parameters of the finite geometry are in good agreement with (4).

Without loss of generality, the solution for the asymmetrically loaded specimen in Fig. 1 can be written as

$$K_{\mathrm{I}}=\frac{6(c-c_0)Q}{W^2}\sqrt{\pi a}F_{\mathrm{I}}(a/W) \tag{5a}$$

$$K_{\mathrm{II}}=\frac{\eta Q}{W^{1/2}}\frac{(a/W)^{3/2}}{(1-a/W)^{1/2}}F_{\mathrm{II}}(a/W) \tag{5b}$$

where, in general, c_0/W and η are functions of a/W, c/W, b_1/W, and b_2/W. The mode I stress intensity factor is not precisely zero where $M=0$, motivating the introduction of c_0. The representation (5) is chosen because it reduces to the reference solution ($c_0/W=0, \eta=1$) when the loading points are sufficiently far from the crack. The finite element results presented below indicate the reference solution is accurate to within about two percent as long as the distance of nearest loading point to the crack is greater than $1.4W$.

Figure 2 displays the dependence of c_0/W on a/W for three values of b_1/W and $\alpha\equiv(b_2-b_1)/W=1$. This was computed as the c/W at which $K_{\mathrm{I}}=0$. If the moment at the crack vanishes (i.e., $c=0$), the mode I factor can be significant when the loading points are near the crack. For example, for the extreme, but not entirely unrealistic case, where $b_1/W=0.6$, $\alpha=1$, $a/W=0.2$, and $c=0$, the mode mixity, $\psi=\tan^{-1}(K_{\mathrm{II}}/K_{\mathrm{I}})$, is 65 deg instead of 90 deg.

Variations of the mode II correction factor η with a/W for several c/W are shown in Fig. 3 for $b_1/W=1.0$ and $\alpha=1$. The error is largest for short cracks and for cracks on the order of a distance W from the closest loading point. Curves corresponding to constant values of the correction factor are plotted in Fig. 4, with $c/W=0.2$ and $\alpha=1$. If the combination ($b_1/W, a/W$) lies above the curve, the correction factor will be smaller than the corresponding η.

Finally, the effect of the parameter $\alpha=(b_2-b_1)/W$ is displayed in Fig. 5 by normalizing each of the respective stress intensity factors by the reference value from (2). These results have been computed with $b_1/W=1.4$ and $c/W=0.2$. The error in the reference values is less than roughly 2 percent when $\alpha>0.5$.

The plots in Figs. 2–5 provide guidance for either: (i) ensuring the test parameters are such that the reference solution (2) can be used with confidence, or (ii) estimating the corrections to the reference solution using (5). As long as the distance between the crack and the nearest loading point is greater than about $1.4W$

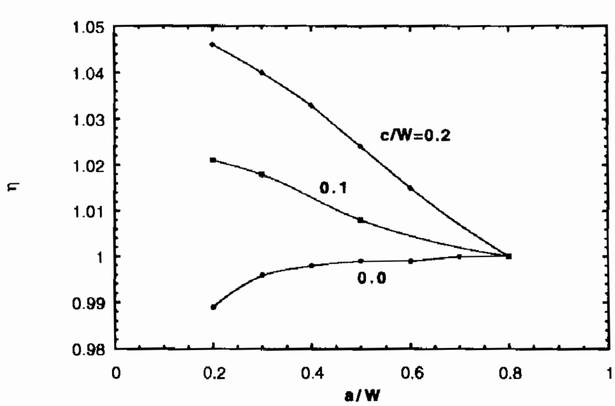

Fig. 3 Correction factor for mode II intensity factor ($\alpha=1$)

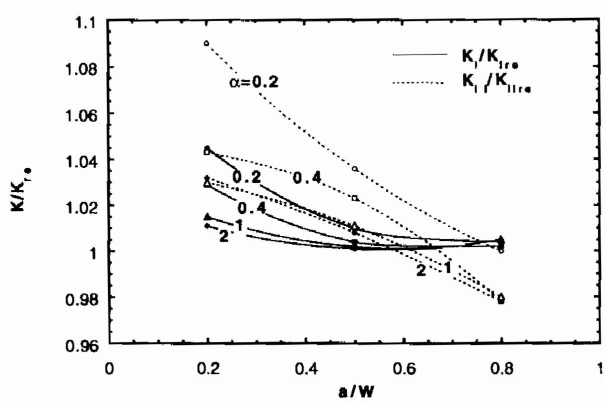

Fig. 5 Role of $\alpha=(b_2-b_1)/W$ in error of the reference solution (2) for $b_1/W=1.4$ and $c/W=0.2$

(i.e., $(b_1-c)/W>1.4$ with $b_2>b_1$) the reference solution is accurate to within a few percent. The errors in the reference solution are the smallest for deep cracks, i.e., $a/W\geq 0.5$.

Acknowledgment

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Large Shearing of a Prestressed Tube

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This study is devoted to a prestressed and hyperelastic tube representing a vascular graft subjected to combined deformations. The analysis is carried out for a neo-Hookean response augmented with unidirectional reinforcing that is characterized by a single additional constitutive parameter for strength of reinforcement. It is shown that the stress gradients can be reduced in presence of prestress. [S0021-8936(00)00101-X]

1 Introduction

Mechanical properties are of major importance when selecting a material for the fabrication of small vascular prostheses. The operation and the handing of prostheses vessel by surgeons, on the one part, the design of such grafts, on the other, induce specific loading and particularly boundary or initial conditions. Consequently, the interest in developing a theoretical model to describe the behavior of the prostheses vessel is proved ([1]). In this paper, we consider a thick-walled prestressed tube, hyperelastic, transversely isotropic, and incompressible assimilated to a vessel graft. We give an exact solution of the stress distributions when the tube is subjected to the simultaneous extension, inflation, torsion, azimuthal, and telescopic shears ([2–10]). The first theoretical results, in the case of a silicone tube, indicate that the increase of prestress minimizes the stress gradients due to the effects of the shear.

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2 Model Formulation

Consider a nonlinearly elastic opened tube defined by the angle ω_0 (Fig. 1). Let us suppose that the tube undergoes two successive deformations; first, including the closure of the tube which induced residual strains ([11]) and second, including inflation, extension, torsion, azimuthal and telescopic shears. The mapping is described by

$$r=r(R)\quad \theta=\left(\frac{\pi}{\omega_0}\right)\omega+\phi\alpha Z+\Theta(r)\quad z=\lambda\alpha Z+\Delta(r)\tag{1}$$

where (R,ω,Z) and (r,θ,z) are, respectively, the reference and the deformed positions of a material particle in a cylindrical system. ϕ is a twist angle per unloaded length, α and λ are stretch ratios (respectively, for the first and the second deformation), Θ is an angle which defined the azimuthal shear, and Δ is an axial displacement which defined the telescopic shear.

It follows from (1) that the physical components of the deformation gradient $\mathbf{F}$ has the following representation in a cylindrical system:

$$\mathbf{F}=\begin{bmatrix} \dot r(R) & 0 & 0\\ r(R)\dot\Theta(r)\dot r(R) & \dfrac{r(R)}{R}\dfrac{\pi}{\omega_0} & r\phi\alpha\\ \dot\Delta(r)\dot r(R) & 0 & \alpha\lambda\end{bmatrix}\tag{2}$$

where the dot denotes the differentiation with respect to the argument.

Incompressibility then requires that $J\equiv\det\mathbf{F}=1$, which upon integration yields

$$r^2=r_i^2+\frac{\omega_0}{\pi\alpha\lambda}(R^2-R_i^2)\tag{3}$$

where R_i and r_i are, respectively, the inner surfaces of the tube in the free and in the loaded configurations (R_e and r_e are the outer surfaces).

The strain energy density per unit undeformed volume for an elastic material, which is locally and transversely isotropic about the $\mathbf{t}(R)$ direction, is given by

$$W=W(I_1,I_2,I_3,I_4,I_5)\tag{4}$$

where

$$I_1=Tr\mathbf{C},\quad I_2=\tfrac{1}{2}[(Tr\mathbf{C})^2-Tr\mathbf{C}^2],\quad I_3=1,$$
$$I_4=\mathbf{tCt},\quad I_5=\mathbf{tC}^2\mathbf{t}\tag{5}$$

are the principal invariants of $\mathbf{C}=\bar{\mathbf{F}}\mathbf{F}$ which is the right Cauchy-Green deformation tensor ($\bar{\mathbf{F}}$ is the transpose of $\mathbf{F}$).

The corresponding response equation for the Cauchy stress σ for transversely isotropic incompressible is (see [12])

$$\sigma=-p\mathbf{1}+2[W_1\mathbf{B}-W_2\mathbf{B}^{-1}+I_4W_4\mathbf{T}\otimes\mathbf{T}$$
$$+\mathbf{I}_4W_5(\mathbf{T}\otimes\mathbf{B}\cdot\mathbf{T}+\mathbf{T}\cdot\mathbf{B}\otimes\mathbf{T})]\tag{6}$$

where $\mathbf{B}=\mathbf{F}\bar{\mathbf{F}}$ is the left Cauchy-Green tensor, $\mathbf{1}$ the unit tensor, and p the unknown hydrostatic pressure associated with the incompressibility constraint, $W_i=(\partial W/\partial I_i)$ $(i=1,2,4,5)$ and $\mathbf{T}=(1/\sqrt{I_4})\mathbf{Ft}$.

From (6), the equilibrium equations in the absence of body forces are reduced to

$$\frac{d\sigma_{rr}}{dr}+\frac{\sigma_{rr}-\sigma_{\theta\theta}}{r}=0\tag{7a}$$